EXTENDED ABSTRACTS

24 th Iranian Algebra Seminar

November 12-13, 2014

Dedicated to Professor Hossein Zakeri

Kharazmi University
Faculty of Mathematical Sciences and Computer
Karaj, Iran

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In the name of God

Good morning President, Vice-chancellors, Distinguished Professors, Participants, Ladies and Gentlemen. I am delighted and honored to have this opportunity to welcome you to 24th Iranian Algebra Seminar. This is an annual seminar established by the Iranian Mathematical Society.

This seminar is dedicated to Professor Hossein Zakeri for his 70th birthday, his distinguished research work, and his retirement. As the seminar secretary, I wish to extend a warm welcome to colleagues from the various Iran universities. We are especially honored this year by the presence of the eminent foreign mathematicians, Professor Shiro Goto from Japan and Professor Santiago Zarzuela from Spain, who have graciously accepted our invitation to be here as the keynote speakers.

I recognize that these sessions are principally designed to enhance the exchange of knowledge and new discoveries in algebra and related fields of applications. These annual gatherings enable the building of a productive dialogue between participants of different nationalities. They also provide an invaluable opportunity for networking and fruitful contacts among international institutions.

Since Kharazmi University is one of the leading universities in mathematics in Iran, this is the second time that the conference is being held as the major host of the event. I am pleased to inform you that as many as over 100 participants from different Iran universities are attending. It is also our pleasure and honor to welcome the esteemed professors who are present to impart their expertise to the meeting.

The seminar organizers would like to use this opportunity to emphasize and to express their appreciations to Professor Zakeri who has made some positive changes in research in general and in particular in commutative algebra. He has trained more than 20 PhDs, whom are now researchers and professors around the country. We evaluate his contributions to the development of the society as admirable. Let us take this opportunity to say to Professor Zakeri: We wish you long life with health.

I would like to take this opportunity to express my gratitude to all delegates and sponsors for their full support, cooperation and contribution to this seminar. I also wish to express my gratitude to the Organizing Committee and the Scientific Committee for their diligence.

Mohammad-Taghi Dibaei
(the seminar secretary)
November 12, 2014

Scientific Committee

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ALMOST GORENSTEIN RINGS
— TOWARDS A THEORY OF HIGHER DIMENSION —

SHIRO GOTO

This lecture is based on the recent work [GTT] jointly with R. Takahashi and N. Taniguchi.

In the previous paper [GMP] the author, N. Matsuoka, and T. T. Phuong gave an alternative definition of one-dimensional almost Gorenstein local rings and developed a basic theory about them. *Almost Gorenstein rings* were originally introduced by V. Barucci and R. Fröberg in the case where the local rings are of dimension one and analytically unramified. They developed in [BF] an excellent theory of almost Gorenstein rings and gave many interesting results. The paper [GMP] aimed at a generalized definition of almost Gorenstein ring, which can be applied to the rings that are not necessarily analytically unramified. One of the purposes of such an alternation is to go beyond a gap in the proof of [BF, Proposition 25] and solve in full generality the problem of when the endomorphism algebra $\mathfrak{m} : \mathfrak{m}$ is a Gorenstein ring, where $\mathfrak{m}$ denotes the maximal ideal in a given Cohen-Macaulay local ring of dimension one.

The present purpose is to search for possible definitions of *higher-dimensional* almost Gorenstein local/graded rings. I will give candidates in terms of embedding of the base Cohen-Macaulay rings into their canonical modules. I shall confirm in my lecture that almost all results of one-dimensional case are safely extended to those of higher dimensional local/graded rings. Examples will be explored and some open problems will be discussed.

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ORDINARY AND SYMBOLIC POWERS OF A HOMOGENEOUS IDEAL: AN ALGEBRO-GEOMETRIC COMPARISON

HASSAN HAGHIGHI

Let I be an ideal in a commutative ring R. Although several basic concepts attached to I such as symmetric algebra, associated graded ring and Rees algebra, are defined in terms of powers of I, there are situations where finer constructions are required. When dealing with associated primes of an ideal, although any power of a maximal ideal $\mathfrak{m} \subset R$ is $\mathfrak{m}$-primary, powers of a non-maximal prime ideal P is not in general P-primary. The P-primary component of the rth power of P, denoted by $P^{(r)}$, is called the *rth symbolic power* of P. Whenever I has a primary decomposition, this definition generalizes for the symbolic powers of the ideal I, as the contraction of $I^r R_S$ to R, where S is the complement of the associated prime ideals of I in R.

Let $I^{(r)}$ be the rth symbolic power of an ideal $0 \neq I \subset R$. There is a considerable research on the comparison of the usual powers of an ideal with its symbolic powers. It follows from definition of symbolic powers that $I^r \subseteqq I^{(r)}$, and in some cases, e.g., when I is a complete intersection, equality holds. For a pair of positive integers (r, m), it is known that $I^r \subseteqq I^{(m)}$ if and only if $m \leq r$. But, less is known on pairs (r, m) such that $I^{(m)} \subset I^r$. In the geometric setting, i.e., when I is a homogenous ideal in $R = \mathbb{K}[x_0, \ldots, x_n]$, it is known that for every positive integer r, $I^{(hr)} \subseteqq I^r$, where h is the height of the ideal I. This result is proved for $\mathrm{char}\mathbb{K} = 0$ by Ein, Lazarsfeld and Smith using multiplier ideal techniques and for $\mathrm{char}\mathbb{K} = p > 0$ by Hochster and Huneke employing tight closure techniques. Since height of any homogeneous ideal in R is at most n, this containment relation implies $I^{(nr)} \subseteqq I^r$ for any positive integer r, and gives a uniform bound for the containment of the symbolic power.

In this talk, some known results on pairs (r, m) for which the inclusion $I^{(m)} \subseteqq I^r$ holds and various criteria which guarantee this inclusion will be reviewed. We recall some existing conjectures on the bounds for r and m such that if r and m lie in these bounds, then the inclusion $I^{(m)} \subseteqq I^r$ holds. These type of relations have many geometric implications. In particular, whenever I is the ideal of a union of linear subspaces of projective space, they give more information about the possible configurations of the subspaces. We use some special configuration of subspaces to test the extent of the validity of these conjectures. In particular, in the special case of a fat points scheme, we provide conditions under which the above inclusion for a pair (r, m) implies the same inclusion for the ideals of schemes which is obtained by further fattening of the given fat points scheme.

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EFFECTS OF NON-NORMAL SUBGROUPS ON THE STRUCTURE OF FINITE GROUPS

HAMID MOUSAVI

My lecture is based on the non-normal subgroups of finite groups and their effects on the structure of finite groups, in particular of finite p-groups. Let G be a finite non-Dedekindian group. We denote the number of conjugacy classes of non-normal subgroups of G, by $\nu(G)$ and the number of conjugacy classes of non-normal cyclic subgroups of G by $\nu_c(G)$. Many authors have worked on $\nu(G)$ and $\nu_c(G)$. For example, in [2, 3, 4, 7, 8, 9, 10] the authors classify the finite p-groups having special values of $\nu(G)$ and in [5, 11, 12, 13] various attempts are made for finding some relations between $\nu(G)$, $\nu_c(G)$ and the nilpotency class of finite nilpotent groups G. Also, the author of [2] investigates the finite non-solvable groups and found all finite non-solvable groups with $\nu(G) < 14$. Finally in [6] the authors find some conditions on $\nu(G)$ such that G is solvable.

Through a historical review of this subject we show that there exists a strong relationship between the structure of G and $\nu(G)$. Then we will state some open problems in the subject, as well as analyzing some open problems stated in [1]. Here are some sort of relevant problems:

(1) Study the non-Dedekindian p-groups G such that $|G : H^G| = p$ or p^2 for all non-normal subgroup H of G, where H^G is the normal closure of H in G.
(2) Classify the p-groups all of whose non-normal abelian subgroups are cyclic.
(3) Classify the non-nilpotent groups all of whose non-normal subgroups are cyclic.

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ON LOCALLY FINITE SIMPLE LIE SUPERALGEBRAS

MALIHE YOUSOFZADEH

Given an arbitrary $n \times n$-matrix A and a subset $\tau \subseteq \{1, \dots, n\}$, one can define the contragredient Lie superalgebra $\mathcal{G}(A, \tau)$ with Cartan matrix A which is presented by a finite set of generators subject to specific relations. Contragredient Lie superalgebras associated to so-called generalized Cartan matrices are known as Kac-Moody Lie superalgebras. These Lie superalgebras are of great importance among contragredient Lie superalgebras; in particular, affine Lie superalgebras i.e., those Kac-Moody Lie superalgebras which are of finite growth, but not of finite dimension and equipped with a nondegenerate invariant even supersymmetric bilinear form, play a significant role in the theory of Lie superalgebras. In the past 40 years, researchers in many areas of mathematics and mathematical physics have been attracted to Kac-Moody Lie superalgebras $\mathcal{G}(A, \emptyset)$ known as Kac-Moody Lie algebras. These Lie algebras are a natural generalization of finite dimensional simple Lie algebras. One of the differences between affine Lie superalgebras and affine Lie algebras is the existence of imaginary roots i.e., roots which are orthogonal to themselves but not to all other roots. In 1990, R. Høegh-Krohn and B. Torresani [3] introduced irreducible quasi simple Lie algebras as a generalization of both affine Lie algebras and finite dimensional simple Lie algebras over complex numbers; see also [1]. The existence of isotropic roots, i.e., roots which are orthogonal to all other roots, is one of the phenomena which occurs in irreducible quasi simple Lie algebras but not in finite dimensional simple Lie algebras. Since 1990, different generalizations of irreducible quasi simple Lie algebras have been studied. Toral type extended affine Lie algebras [2], locally extended affine Lie algebras [5] and invariant affine reflection algebras [4], as a generalization of the last two stated classes, are examples of these generalizations.

Basic classical simple Lie superalgebras, orthosymplectic Lie superalgebras of arbitrary dimension as well as specific extensions of particular root graded Lie superalgebras satisfy certain properties which in fact are the super version of the axioms defining invariant affine reflection algebras. This gives a motivation to study extended affine Lie superalgebras [6] which are Lie superalgebras satisfying these certain properties. Extended affine Lie superalgebras are generalization of both affine Lie superalgebras and invariant affine reflection algebras. Roughly speaking, an extended affine Lie superalgebra is a Lie superalgebra having a weigh space decomposition with respect to a nontrivial abelian subalgebra of the even part and equipped with a nondegenerate invariant even supersymmetric bilinear form such that the weight vectors associated to so-called real roots are ad-nilpotent. We call split simple extended affine Lie superalgebras locally finite basic classical Lie superalgebras. We study the structure of

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locally finite basic classical Lie superalgebras; specially, we show that a locally finite basic classical Lie superalgebra is a direct union of finite dimensional basic classical Lie subsuperalgebras. We finally give the classification of locally finite basic classical Lie superalgebras.

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ON THE DIVISORS OF A MODULE, THEIR REES ALGEBRAS AND BLOW UP

SANTIAGO ZARZUELA

Paraphrasing W. V. Vasconcelos, the divisors of a (finitely generated) module E over a commutative ring R are ideals of R that carry important information about the structure and properties of E. Typical examples are the Fitting ideals or the determinant. In this talk we shall relate these classical divisors to some more recently introduced ones like the norm of E (as defined by O. Villamayor) or the (generic) Bourbaki ideal (as defined by A. Simis, B. Ulrich and W. V. Vasconcelos). Both are attached to blow up constructions: in the first case to the blow up at E and in the second one to the so named Rees algebra of E. The relationships among all these divisors allow to understand in a better way some of the universal properties with respect to E of these blow ups.

This is a joint work with Ana L. Branco Correia.

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NON-SOLVABLE GROUPS GENERATED BY INVOLUTIONS IN WHICH EVERY INVOLUTION IS LEFT 2-ENGEL

ALIREZA ABDOLLAHI

Abstract. The following problem is proposed as Problem 18.57 in [The Kourovka Notebook, No. 18, 2014] by D. V. Lytkina: Let G be a finite 2-group generated by involutions in which $[x, u, u] = 1$ for every $x \in G$ and every involution $u \in G$. Is the derived length of G bounded?

The question is asked of an upper bound on the solvability length of finite 2-groups generated by involutions in which every involution (not only the generators) is also left 2-Engel. We negatively answer the question.

1. Introduction and Result

The following problem is proposed as Problem 18.57 of [2] by D. V. Lytkina:

Question 1.1. Let G be a finite 2-group generated by involutions in which $[x, u, u] = 1$ for every $x \in G$ and every involution $u \in G$. Is the derived length of G bounded?

Question 1.1 is asked of an upper bound on the solvability length of finite 2-groups generated by involutions in which all involutions of groups (not only the generators) are also left 2-Engel elements. We negatively answer the question. In the proof we need some well-known facts about the groups of exponent 4.
It is know that groups of exponent 4 are locally finite [6] and the free Burnside group $\mathfrak{B}$ of exponent 4 with infinite countable rank is not solvable [5]. In [1], it is proved that the solvability of $\mathfrak{B}$ is equivalent to the one of the group H defined as follows: Let H be the freest group generated by elements $\{x_i \mid i \in \mathbb{N}\}$ with respect to the following relations:
(1) $x_i^2 = 1$ for all $i \in \mathbb{N}$;
(2) the normal closure $\langle x_i \rangle^H$ is abelian for all $i \in \mathbb{N}$;
(3) $h^4 = 1$ for all $h \in H$.

Therefore H is a non-solvable group of exponent 4 generated by involutions x_i ($i \in \mathbb{N}$). Note that the relation (2) above is equivalent to say that x_i is a left 2-Engel element of H that is $[x, x_i, x_i] = 1$ for all $x \in H$. We do not know if H has the property requested in Question 1.1, that is, whether every involution $u \in H$ is a left 2-Engel element of H. Instead we find a quotient of H which is still non-solvable but it satisfies the latter property. The latter quotient of H will provide a counterexample

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for Question 1.1. To introduce the quotient we need to recall some definitions and results on right 2-Engel elements.
For any group G, $R_2(G)$ denotes the set of all right 2-Engel elements of G, i.e.

$$R_2(G) = \{a \in G \mid [a, x, x] = 1 \text{ for all } x \in G\}.$$

It is known [3] that $R_2(G)$ is a characteristic subgroup of G. The subgroup $R_2(G)$ is a 2-Engel group that is $[x, y, y] = 1$ for all $x, y \in R_2(G)$. Thus $R_2(G)$ is nilpotent of class at most three [4] and so it is of solvable length at most 2.

Theorem 1.2. *Let H be the freest group defined above. Then $\overline{H} = H/R_2(H)$ satisfies the following condition:*
(4) *all involutions $u \in \overline{H}$ are left 2-Engel in $\overline{H}$.*
Furthermore, $\overline{H}$ is not solvable so that there is no upper bound on solvability lengths of finite 2-groups $\overline{H}_n = \frac{\langle x_1,\dots,x_n\rangle}{R_2(\langle x_1,\dots,x_n\rangle)}$ which satisfy all conditions (1), (2), (3) and (4) above.

2. **Proof of Theorem 1.2**

The following is the key lemma of the paper.

Lemma 2.1. *Let G be any group of exponent 4 and $b \in G$ be such that $b^2 \in R_2(G)$. Then $[a, b, b] \in R_2(G)$ for all $a \in G$. This means that, in every group G of exponent 4, every involution of the quotient $G/R_2(G)$ has an abelian normal closure.*

Proof. Let N be the freest group generated by elements a, b, c subject to the following relations:
(1) $x^4 = 1$ for all $x \in N$;
(2) $[b^2, x, x] = 1$ for all $x \in N$.
By [6] it is known that N is finite. Now by nq package [7] one can construct N in GAP [8] by the following commands:

```
LoadPackage("nq"); F:=FreeGroup(4); a:=F.1;b:=F.2;c:=F.3;x:=F.4;
G:=F/[x^4,LeftNormedComm([b^2,x,x])]; N:=NilpotentQuotient(G,[x]);
gen:=GeneratorsOfGroup(G,[x]);
LeftNormedComm([gen[1],gen[2],gen[2],gen[3],gen[3]]);
```

Note that in above `gen[1]`, `gen[2]` and `gen[3]` correspond to the free generators a, b and c, respectively. The output of last command in above (which is `id` the trivial element of N) shows that $[a, b, b, c, c] = 1$. This completes the proof. □

It may be interesting in its own right that the group N defined in the proof of Lemma 2.1 is nilpotent of class 7 and order 2^{41}.

Proof of Theorem 1.2. It follows from Lemma 2.1 that $\overline{H} = H/R_2(H)$ has the property (4) mentioned in the statement of Theorem 1.2. Since H is not solvable by [1] and [5], it follows that there is no upper bound on the solvable lengths of finite 2-groups $\overline{H}_n$. By construction $\overline{H}_n$ is generated by involutions and by Lemma 2.1 all involutions in $\overline{H}_n$ are left 2-Engel. This completes the proof. □

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z-FILTERS AND ZERO SETS IN POINTFREE TOPOLOGY

M. ABEDI†

Abstract. In this talk, zero sets in pointfree topology are defined. The basic relations that we expect from zero sets are studied and it is shown that the family of all zero sets is a base for the collection of closed sets. Also, z-filters are introduced in terms of pointfree topology. Then the relationship between z-filters and ideals, particularly maximal ideals, is examined.

This is a joint work with A.A. Estaji‡ and A. Karimi Feizabadi††.

1. Introduction

To study the ring $C(X)$, X is a topological space, zero sets and z-ideals have an important role (for more details see [4]). Banaschewski and Gilmour studied the ring $C(L)$ of real-valued continuous functions on frame L as the pointfree version of $C(X)$ and took cozero elements as pointfree version of cozero sets (1996, [2]).

In this talk, by considering prime elements of a given frame L as pointfree points of L, we define the trace of an element α of $C(L)$ on any point p of L that is a real number denoted by $\alpha[p]$, and also, zero set of α by $Z(\alpha) = \{p \in \Sigma L : \alpha[p] = 0\}$ is defined. The real number $\alpha[p]$ is defined by Dedekind cut $(L(p, \alpha), U(p, \alpha))$, where $L(p, \alpha) = \{r \in \mathbb{Q} : \alpha(-, r) \leq p\}$ and $U(p, \alpha) = \{s \in \mathbb{Q} : \alpha(s, -) \leq p\}$ (Proposition 2.1). Also, the map $\widetilde{p} : C(L) \to \mathbb{R}$ given by $\widetilde{p}(\alpha) = \alpha[p]$ is an f-ring homomorphism (Proposition 2.3).

A relation between zero sets and cozero elements is given in Lemma 3.2, that is: $p \in Z(\alpha)$ if and only if $coz(\alpha) \leq p$. The basic relations that we expect from zero sets are shown in Proposition 3.3. The property of the family of zero sets which is a base for the closed sets is proved in Proposition 3.5 for completely regular frames. Moreover, if L is a spatial frame and the family $Z[L]$ of all zero sets is a base for the closed sets of ΣL, then L is completely regular.

Some natural relations between ideals of $C(L)$ and z-filters that are filters of the lattice of $Z[L]$ are explained in the last section. Also, we seek some relations among z-ultrafilters, maximal ideals and families with finite intersection property.

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2. Preliminaries

Here, we recall some definitions and results from the literature on frames and the pointfree version of the ring of continuous real valued functions. For further information concerning frame-theoretic and pointfree function rings, see [5]and [1], respectively.

A *frame* is a complete lattice L in which the distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s : s \in S\}$$

holds for all $x \in L$ and $S \subseteq L$. We denote the top element and the bottom element of L by $\top$ and $\perp$ respectively.

An element $p \in L$ is called *prime* if $p < \top$ and $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. Recall that the contravariant functor Σ from **Frm** to the category **Top** of topological spaces assigns to each frame L its *spectrum* ΣL of prime elements with $\Sigma_a = \{p \in \Sigma L | a \not\leq p\}$ $(a \in L)$ as its open sets. Also, for a frame map $h : L \to M$, $\Sigma h : \Sigma M \to \Sigma L$ takes $p \in \Sigma M$ to $h_*(p) \in \Sigma L$, where $h_* : M \to L$ is the *right adjoint* of h characterized by the condition $h(a) \leq b$ if and only if $a \leq h_*(b)$ for all $a \in L$ and $b \in M$. Note that h_* preserves primes and arbitrary meets.

The *cozero map* is the map $coz : C(L) \to L$, defined by

$$coz(\alpha) = \alpha(-, 0) \vee \alpha(0, -).$$

For $A \subseteq C(L)$, let $Coz(A) = \{coz(\alpha) : \alpha \in A\}$ with the cozero part of a frame L, $Coz(C(L))$, called $CozL$ by previous authors. It is known that L is completely regular if and only if $Coz(C(L))$ generates L. For more details about *cozero map* and its properties which are used in this note see [1].

Here we recall three necessary propositions from [3].

Proposition 2.1. [3] Let L be a frame. If $p \in \Sigma L$ and $\alpha \in C(L)$, then $(L(p, \alpha), U(p, \alpha))$ is a Dedekind cut for a real number which is denoted by $\widetilde{p}(\alpha)$.

Proposition 2.2. [3] If p is a prime element of a frame L, then there exists a unique map $\widetilde{p} : C(L) \longrightarrow \mathbb{R}$ such that for each $\alpha \in C(L)$, $r \in L(p, \alpha)$, and $s \in U(p, \alpha)$ we have $r \leq \widetilde{p}(\alpha) \leq s$.

Proposition 2.3. [3] If p is a prime element of frame L, then $\widetilde{p} : C(L) \longrightarrow \mathbb{R}$ is an onto f-ring homomorphism. Also, $\widetilde{p}$ is a linear map with $\widetilde{p}(\mathbf{1}) = 1$.

Let p be a prime element of L. Throughout this paper for every $\alpha \in C(L)$ we define $\alpha[p] = \widetilde{p}(\alpha)$.

Definition 2.4. A frame L is called *weakly spatial*, if $a < \top$ implies $\Sigma_a \neq \Sigma_\top$.

3. Zero set

We introduce the pointfree version of the zero set of $f \in C(X)$ given by $Z(f) = \{x \in X : f(x) = 0\}$. In poinfree version, we use prime elements $p \in L$ to replace points $x \in X$.

Definition 3.1. Let $\alpha \in C(L)$. We define

$$Z(\alpha) = \{p \in \Sigma L : \alpha[p] = 0\}.$$

This set is called a zero-set in L. For $A \subseteq C(L)$, we write $Z[A]$ to designate the family of zero-sets $\{Z(\alpha) : \alpha \in A\}$. The family $Z[C(L)]$ of all zero-sets in L will also be denoted, for simplicity, by $Z[L]$.

Lemma 3.2. *Let p be a prime element of frame L. For $\alpha \in C(L)$, $\alpha[p] = 0$ if and only if $coz(\alpha) \leq p$. Hence $Z(\alpha) = \Sigma L - \Sigma_{coz(\alpha)}$.*

Yhe above lemma has an important role in describing the zero sets. Now, with the aid of this lemma, we obtain the basic relations that we expect from zero sets.

Proposition 3.3. *For every $\alpha, \beta \in C(L)$, we have*

(1) *For every $n \in \mathbb{N}$, $Z(\alpha) = Z(|\alpha|) = Z(\alpha^n)$.*
(2) *$Z(\alpha) \cap Z(\beta) = Z(|\alpha| + |\beta|) = Z(\alpha^2 + \beta^2)$.*
(3) *$Z(\alpha) \cup Z(\beta) = Z(\alpha\beta)$.*
(4) *If α is a unit of $C(L)$, then $Z(\alpha) = \emptyset$.*
(5) *$Z[L]$ is closed under countable intersection.*

In Proposition 3.4, we show that every pointfree zero set $Z(\alpha)$, $\alpha \in C(L)$ is equal to a topological zero set $Z(f)$ for some $f : \Sigma L \to \mathbb{R}$. But before that we need some necessary tools.

There is a homeomorphism $\tau : \Sigma\Re \to \mathbb{R}$ such that $r < \tau(p) < s$ if and only if $(r, s) \not\leq p$ for all prime elements p of $\Re$ and all $r, s \in \mathbb{Q}$ (see Proposition 1 of [2], page 12).

Proposition 3.4. *Let L be a frame and $\alpha \in C(L)$. Then $Z(\alpha) = Z(\tau \circ \Sigma\alpha)$.*

Now, we determine when the family $Z[L]$ is a base for the closed sets of ΣL .

Proposition 3.5. *For each frame L, the following statements hold:*

(1) *If L is a completely regular frame, then the family $Z[L]$ of all zero sets is a base for the closed sets of ΣL.*
(2) *If L is a spatial frame and the family $Z[L]$ of all zero sets is a base for the closed sets of ΣL, then L is completely regular.*

Question 3.6. Is there a frame L which is neither sapatial nor completely regular such that $Z[L]$ is a basis for the closed sets of ΣL?

4. Ideals and z-filters

Continuing our study of the relations between algebraic properties of $C(L)$ and lattice properties of L, we now examine the special features of the family of zero-sets of an ideal of functions. Such a family possess properties analogous to those of a filter; this fact will play a central role in the development.

Definition 4.1. A nonempty subfamily $\mathcal{F}$ of $Z[L]$ is called a *z-filter* on L provided that

(1) $\emptyset \notin \mathcal{F}$,
(2) if $Z_1, Z_2 \in \mathcal{F}$, then $Z_1 \cap Z_2 \in \mathcal{F}$, and
(3) if $Z \in \mathcal{F}$, $Z' \in Z[L]$, and $Z \subseteq Z'$, then $Z' \in \mathcal{F}$.

By (3), ΣL belongs to every z-filter.

Let $\mathcal{A}$ be a nonempty family of sets. $\mathcal{A}$ is said to have the finite intersection property provided that the intersection of any finite number of members of $\mathcal{A}$ is nonempty.

Every family $\mathcal{B}$ of zero-sets of a frame L that has the finite intersection property is contained in a z-filter. Also,

$$\mathcal{F} = \{Z \in Z[L] : \text{ there exists a finite subset } \mathcal{A} \text{ of } \mathcal{B} \text{ such that } \bigcap \mathcal{A} \subseteq Z\}$$

is the smallest z-filter containing $\mathcal{B}$.

Proposition 4.2. *For each frame L, the following statements hold:*

(1) *If L is a weakly spatial frame and I is a proper ideal in $C(L)$, then the family $Z[I] = \{Z(\alpha) : \alpha \in I\}$ is a z-filter on L.*
(2) *If $\mathcal{F}$ is a z-filter on L, then the family $Z^{\leftarrow}[\mathcal{F}] = \{\alpha : Z(\alpha) \in \mathcal{F}\}$ is a proper ideal in $C(L)$.*

Remark 4.3. By a z-ultrafilter on L is meant a maximal z-filter, i.e., one not contained in any other z-filter. Thus, for every L, the following statements hold:

(1) A z-ultrafilter is a maximal subfamily of $Z[L]$ with the finite intersection property.
(2) Every subfamily of $Z[L]$ with the finite intersection property is contained in some z-ultrafilter.

Proposition 4.4. *Let L be a weakly spatial frame.*

(1) *If M is a maximal ideal of $C(L)$, then $Z[M]$ is a z-ultrafilter on L.*
(2) *If $\mathcal{F}$ is a z-ultrafilter on L, then $Z^{\leftarrow}[\mathcal{F}]$ is a maximal ideal of $C(L)$.*

Corollary 4.5. *Let L be a weakly spatial frame.*

(1) *Let M be a maximal ideal of $C(L)$. If $Z(\alpha)$ meets every member of $Z[M]$, then $\alpha \in M$.*
(2) *Let $\mathcal{F}$ be a z-ultrafilter on L. If a zero-set Z meets every member of $\mathcal{F}$, then $Z \in \mathcal{F}$.*

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OPTIMAL SYSTEM OF SUBGROUPS AND CLASSIFICATION OF INVARIANT SOLUTIONS FOR THE (2+1)-DIMENSIONAL ZK-MEW EQUATION

FATEMEH AHANGARI

Abstract. Lie group analysis, based on symmetry and invariance principles, can be regarded as the only powerful systematic procedure for solving nonlinear differential equations analytically. In this talk, a detailed analysis of a significant nonlinear model system, the two dimensional ZK-MEW equation, is presented via Lie groups. For this purpose, the Lie group of symmetries is computed and the algebraic structure of the Lie algebra of symmetries is comprehensively discussed. It is proved that the Lie algebra of symmetries is solvable and non-semisimple and admits a three dimensional nilpotent solvable nilradical. Furthermore, the optimal system of symmetry subgroups is constructed and a complete classification of the group invariant solutions is determined.

1. Introduction

By the late nineteenth century, Sophus Lie initiated his studies on continuous groups (Lie groups) with the aim of putting order and thereby extending systematically, the techniques for solving differential equations. According to Lie's fundamental theory, the problem of determining the Lie group of point transformations leaving invariant a differential equation (ordinary or partial) is reduced to solving the related linear systems of determining equations for its infinitesimal generators [1]. This group in the Lie's theory consists in geometric transformations which act on the set of solutions by transforming their graphs. A lot of properties both of the system and of their solutions can be implied by the knowledge of the symmetry group. Determining the group invariant solutions, construction of new solutions for the system from the known ones, classification of the group invariant solutions, reduction of the order of ordinary differential equations, detection of linearizing transformations and mapping solutions to other solutions are the other important applications of Lie groups in the theory of differential equations. For many other applications of Lie symmetries refer to [2, 3, 4]. It is worth mentioning that a large number of equations in many areas of physics, applied mathematics and engineering appear as nonlinear wave equations. One of the most significant is one dimensional non-linear wave equations is the KdV equation which describes the evolution of weakly non-linear and weakly dispersive wave which is applied to various fields such as fluid physics, plasma physics, solid state physics and

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quantum field theory. One of the significant famous two dimensional generalizations of the KdV equation is known as the ZK-MEW equation and defined as follows [5]:

$$u_t + a(u^3)_x + (bu_{xt} + ru_{yy})_x = 0. \tag{1}$$

Recently, much efforts have been devoted on the construction of exact traveling wave solutions of these types of equations and a lot of powerful methods have been proposed, e.g. Backlund transform, Jacobi elliptic function expansion method, sine-cosine method and inverse scattering transform. The main purpose of this paper is to obtain some new types of exact solutions for the ZK-MEW equation by applying the powerful Lie group method.

The structure of my talk, is as follows: In section 2, using the basic Lie group method, the most general Lie point symmetry group of the ZK-MEW equation is determined. In section 3, some results obtained from the algebraic structure of the Lie algebra of symmetries are given. Section 4 is devoted to construction of the optimal system of symmetry subgroups and complete classification of the corresponding invariant solutions. I end my talk by conclusion.

2. Computation of the Lie Algebra of Symmetries

In this section, we will perform Lie group method for Equation (1). Firstly, let us consider a one-parameter Lie group of infinitesimal transformation:

$$\begin{aligned} \bar{x} &= x + \varepsilon\zeta^1(x,y,t,u) + O(\varepsilon^2), \quad \bar{y} = y + \varepsilon\zeta^2(x,y,t,u) + O(\varepsilon^2), \\ \bar{t} &= t + \varepsilon\zeta^3(x,y,t,u) + O(\varepsilon^2), \quad \bar{u} = u + \varepsilon\Omega(x,y,t,u) + O(\varepsilon^2), \end{aligned}$$

with a small parameter $\varepsilon << 1$. The symmetry generator associated with the above group of transformations can be written as:

$$\mathbf{V} = \zeta^1(x,y,t,u)\frac{\partial}{\partial x} + \zeta^2(x,y,t,u)\frac{\partial}{\partial y} + \zeta^3(x,y,t,u)\frac{\partial}{\partial t} + \Omega(x,y,t,u)\frac{\partial}{\partial u}. \tag{2}$$

The third prolongation of $\mathbf{V}$ is the vector field

$$\begin{aligned} \mathbf{V}^{(3)} = \mathbf{V} \ &+ \ \Omega^x\frac{\partial}{\partial u_x} + \Omega^y\frac{\partial}{\partial u_y} + \Omega^t\frac{\partial}{\partial u_t} + \Omega^{2x}\frac{\partial}{\partial u_{2x}} \\ &+ \ \Omega^{xy}\frac{\partial}{\partial u_{xy}} + \Omega^{xt}\frac{\partial}{\partial u_{xt}} + \Omega^{2y}\frac{\partial}{\partial u_{2y}} + \Omega^{yt}\frac{\partial}{\partial u_{yt}} + \Omega^{2t}\frac{\partial}{\partial u_{2t}} \\ &+ \ \Omega^{3x}\frac{\partial}{\partial u_{3x}} + \cdots + \Omega^{3y}\frac{\partial}{\partial u_{3y}} + \Omega^{3t}\frac{\partial}{\partial u_{3t}} + \cdots \end{aligned} \tag{3}$$

with coefficients $\Omega^J = D_J(\Omega - \sum_{i=1}^3 \zeta^i u_i^\alpha) + \sum_{i=1}^3 \zeta^i u_{J,i}$, where $J = (i_1, \ldots, i_k)$, $1 \le i_k \le 3$, $1 \le k \le 4$ and the sum is over all $J's$ of order $0 < \#J \le 4$.
By theorem (6.5) in [4], the invariance condition for the ZK-MEW equation is given by the relation:

$$\mathbf{V}^{(3)}[u_t + 3au_xu^2 + bu_{txx} + ru_{xxy}] = 0 \tag{4}$$

The invariance condition (4) is equivalent with the following equation:

$$\Omega^t + 3a\Omega^x\Omega^2 + b\Omega^{txx} + r\Omega^{xxy} = 0 \tag{5}$$

Substituting the corresponding coefficients into invariance condition (5), we are left with a polynomial equation involving the various derivatives of $u(x,y,t)$ whose coefficients are certain derivatives of ζ^1, ζ^2, ζ^3 and Ω. Since, ζ^1, ζ^2, ζ^3 and Ω depend only on (x,y,t,u), we can equate the individual coefficients to zero. Hence, by solving the resulted set of determining equations, it is deduced that:

Theorem 2.1. *The Lie group of point symmetries of the ZK-MEW equation has a Lie algebra generated by the vector fields* $\mathbf{V} = \zeta^1\frac{\partial}{\partial x} + \zeta^2\frac{\partial}{\partial z} + \zeta^3\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial u}$, *where*

$$\begin{aligned} \zeta^1(x,y,t,u) &= -\frac{xr}{2b}c_2 + c_5, \qquad \zeta^2(x,y,t,u) = \frac{yr}{2b}c_2 - 2c_1y + c_4, \\ \zeta^3(x,y,t,u) &= -\frac{2t\left(\frac{1}{4}c_2r + bc_1\right)}{b} + c_2y + c_3, \qquad \Omega(x,y,t,u) = c_1u. \end{aligned}$$

and $c_i,\ i = 1, ..., 5$ *are arbitrary constants.*

Corollary 2.2. *The infinitesimal generators of every one parameter Lie group of point symmetries of the ZK-MEW equation are:*

$$\begin{aligned} &\mathbf{V}_1 = \frac{\partial}{\partial x}, \quad \mathbf{V}_2 = \frac{\partial}{\partial y}, \quad \mathbf{V}_3 = \frac{\partial}{\partial t}, \quad \mathbf{V}_4 = 2x\frac{\partial}{\partial x} - 4y\frac{\partial}{\partial y} - \frac{4yb}{r}\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \\ &\mathbf{V}_5 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - \frac{-rt + 2yb}{r}\frac{\partial}{\partial t}. \end{aligned}$$

The nonzero commutators of symmetry generators of the ZK-MEW equation are given as follows:

$$\begin{aligned} &[V_1, V_4] = 2V_1, \quad [V_1, V_5] = V_1, \quad [V_2, V_4] = -4V_2 - \frac{4b}{r}V_3, \\ &[V_2, V_5] = -V_2 - \frac{2b}{r}V_3, \quad [V_3, V_5] = V_3. \end{aligned}$$

3. Algebraic Structure of the Lie algebra of Symmetries

In this part, we determine the structure of symmetry Lie algebra of the ZK-MEW equation which is denoted by $\mathtt{g}$. It is proved that $\mathtt{g}$ has no non-trivial *Levi decomposition* in the form $\mathtt{g} = \mathtt{r} \ltimes \mathtt{g}_1$, because $\mathtt{g}$ has not any non-trivial radical, i.e. if $\mathtt{r}$ be the radical of $\mathtt{g}$, then $\mathtt{g} = \mathtt{r}$.

The Lie algebra $\mathtt{g}$ is solvable and non-semisimple. It is solvable, because if $\mathtt{g}^{(1)} = < \mathtt{V_i}, [\mathtt{V_i}, \mathtt{V_j}] > = [\mathtt{g}, \mathtt{g}]$, we have:

$$\mathtt{g}^{(1)} = [\mathtt{g}, \mathtt{g}] = < 2\mathtt{V}_1, -4\mathtt{V}_2 - \frac{4\mathtt{b}}{\mathtt{r}}\mathtt{V}_3, -\mathtt{V}_2 - \frac{2\mathtt{b}}{\mathtt{r}}\mathtt{V}_3 >,$$

and

$$\mathtt{g}^{(2)} = [\mathtt{g}^{(1)}, \mathtt{g}^{(1)}] = 0.$$

so we have the following chain of ideals $\mathtt{g}^{(1)} \supset \mathtt{g}^{(2)} = \{0\}$. Also, $\mathtt{g}$ is not semisimple, because its killing form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 6 \\ 0 & 0 & 0 & 6 & 3 \end{pmatrix}$$

is degenerate. Taking into account the table of commutators, $\mathfrak{g}$ has a three dimensional nilradical which is nilpotent and solvable and spanned by $< V_1, V_2, V_3 >$.

4. Optimal System of Symmetry Subgroups and Classification of the Invariant Solutions

Let G be the symmetry Lie group of the ZK-MEW equation. Now, G operates on the set of the solutions of equation (1) denoted by S. Let H and $\tilde{H}$ be two connected, $r-$dimensional Lie subgroups of the Lie group G with corresponding Lie subalgebras $\mathtt{h}$ and $\tilde{\mathtt{h}}$ of the Lie algebra $\mathfrak{g}$ of G. Then according to [2, 3], $\tilde{H} = gHg^{-1}$ are conjugate subgroups if and only if $\tilde{\mathtt{h}} = \mathrm{Ad}(g) \cdot \mathtt{h}$ are conjugate subalgebras. Thus, the problem of determining an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. So we concentrate on it in the following.
Each $V_i,\ i = 1, ..., 5$, of the basis symmetries generates an adjoint representation (or inner automorphism) $\mathrm{Ad}(\exp(\varepsilon V_i))$ defined by the Lie series

$$\mathrm{Ad}(\exp(\varepsilon.V_i).V_j) = V_j - \varepsilon.[V_i, V_j] + \frac{\varepsilon^2}{2}.[V_i, [V_i, V_j]] - \cdots \tag{6}$$

where $[V_i, V_j]$ is the commutator for the Lie algebra, ε is a parameter, and $i, j = 1, \cdots, 5$ ([3]). In table 1 all the adjoint representations of the ZK-MEW Lie group, with the (i, j) entry indicating $\mathrm{Ad}(\exp(\varepsilon V_i))V_j$, are given. Let $F_i^s : \mathfrak{g} \to \mathfrak{g}$ defined by

Table 1. Adjoint representation generated by the basis symmetries of the ZK-MEW Lie algebra

Ad	V_1	V_2	V_3	V_4	V_5
V_1	V_1	V_2	V_3	$-2sV_1 + V_4$	$-sV_1 + V_5$
V_2	V_1	V_2	V_3	$4sV_2 + \frac{4sb}{r}V_3 + V_4$	$sV_2 + \frac{2sb}{r}V_3 + V_5$
V_3	V_1	V_2	V_3	V_4	$-sV_3 + V_5$
V_4	$e^{2s}V_1$	$e^{-4s}V_2 - \frac{b(1-e^{4s})}{r}V_3$	V_3	V_4	V_5
V_5	$e^{s}V_1$	$e^{-s}V_2 + \frac{-b}{r}(e^{s} - e^{-s})V_3$	$e^{s}V_3$	V_4	V_5

$V \mapsto \mathrm{Ad}(\exp(s_i V_i).V)$ is a linear map, for $i = 1, \cdots, 5$. The matrix M_i^s of F_i^s with respect to basis $\{V_1, \cdots, V_5\}$ is

$$M_1^s = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -2s & 0 & 0 & 1 & 0 \\ -s & 0 & 0 & 0 & 1 \end{pmatrix} \quad M_2^s = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 4s & \frac{4sb}{r} & 1 & 0 \\ 0 & s & \frac{2sb}{r} & 0 & 1 \end{pmatrix} \quad M_3^s = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -s & 0 & 1 \end{pmatrix}$$

$$M_4^s = \begin{pmatrix} \exp(2s) & 0 & 0 & 0 & 0 \\ 0 & \exp(-4s) & -\frac{b(1-\exp(4s))}{r} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_5^s = \begin{pmatrix} \exp(4s) & 0 & 0 & 0 & 0 \\ 0 & \exp(-4s) & -\frac{b(\exp(s)-\exp(-s))}{r} & 0 & 0 \\ 0 & 0 & \exp(s) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We can expect to simplify the following arbitrary element,

$$V = \vartheta_1 V_1 + \vartheta_2 V_2 + \vartheta_3 V_3 + \vartheta_4 V_4 + \vartheta_5 V_5. \tag{7}$$

of the ZK-MEW Lie algebra $\mathfrak{g}$. Note that the elements of $\mathfrak{g}$ can be represented by vectors $\vartheta = (\vartheta_1, ..., \vartheta_5) \in \mathbb{R}^5$ since each of them can be written in the form (7) for some constants $\vartheta_1, ..., \vartheta_5$. Hence, the adjoint action can be regarded as (in fact is) a group of linear transformations of the vectors $(\vartheta_1, ..., \vartheta_5)$.
Therefore, the following theorem can be stated:

Theorem 4.1. *The one-dimensional optimal system of the ZK-MEW Lie algebra* $\mathfrak{g}$ *is given by*

$$(1):\ V_1 + \alpha V_3 = \frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial t},$$
$$(2):\ V_1 + \alpha V_2 + \beta V_5 = (1+\beta x)\frac{\partial}{\partial x} + (\alpha - \beta y)\frac{\partial}{\partial y} - \beta\frac{(-rt+2yb)}{r}\frac{\partial}{\partial t},$$
$$(3):\ V_3 + \alpha V_4 = 2x\alpha\frac{\partial}{\partial x} - 4\alpha y\frac{\partial}{\partial y} + (1 - \frac{4\alpha yb}{r})\frac{\partial}{\partial t} + \alpha u\frac{\partial}{\partial u},$$
$$(4):\ V_2 + \alpha V_5 = x\alpha\frac{\partial}{\partial x} + (1-\alpha y)\frac{\partial}{\partial y} - \alpha\frac{(-rt+2yb)}{r}\frac{\partial}{\partial t},$$
$$(5):\ \alpha V_4 + \beta V_5 = (2x\alpha + x\beta)\frac{\partial}{\partial x} - (4\alpha y + y\beta)\frac{\partial}{\partial y} - (\frac{4yb\alpha}{r} + \frac{(-rt+2yb)}{r}\beta)\frac{\partial}{\partial t} + \alpha u\frac{\partial}{\partial u}.$$

where $\alpha, \beta \in \mathbb{R}$ *and* $\alpha \neq 0$.

Conclusion

In this talk the criterion of invariance of the equation under the infinitesimal prolonged infinitesimal generators is applied in order to determine the most general Lie point symmetries group of a well known nonlinear dynamical system: two dimensional ZK-MEW equation. The algebraic structure of $\mathfrak{g}$, the Lie algbera of symmetries of the analyzed model is discussed and it is proved that $\mathfrak{g}$ is a solvable, non-semisimple algebra and admits a three dimensional nilpotent and solvable nilradical. Mainly, the optimal system of symmetry subgroups is constructed which leads to the preliminary classification of the group invariant solutions. For this purpose, in this paper, it is concentrated on this significant fact that the problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. This tends to determine a list of conjugacy inequivalent subalgebras with the property that any other subalgebra is equivalent to a unique member of the list under some element of the adjoint representation for some element of a considered Lie group.

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REGULARITY OF FILTERED MODULES AND LINEARITY DEFECT

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Abstract. Given a finitely generated module M over a commutative local ring (or a standard graded k-algebra) $(R, \mathfrak{m}, k)$ we detect its complexity in terms of numerical invariants coming from suitable $\mathfrak{m}$-stable filtrations $\mathbb{M}$ on M. We study the Castelnuovo-Mumford regularity of $gr_{\mathbb{M}}(M)$ and the linearity defect of M, denoted $\mathrm{ld}_R(M)$, through a deep investigation based on the theory of standard bases. If M is a graded R-module, then $\mathrm{reg}_R(gr_{\mathbb{M}}(M)) < \infty$ implies $\mathrm{reg}_R(M) < \infty$ and the converse holds provided M is of homogenous type. An analogous result can be proved in the local case in terms of the linearity defect. Motivated by a positive answer in the graded case, we present for local rings a partial answer to a question raised by Herzog and Iyengar of whether $\mathrm{ld}_R(k) < \infty$ implies R is Koszul.

1. Introduction

Throughout this talk $(R, \mathfrak{m}, k)$ is a commutative Noetherian local ring (or a standard graded k-algebra) with maximal ideal (or homogeneous maximal ideal) $\mathfrak{m}$ and residue field k. All the modules we consider are finitely generated over R. Let M be an R-module which is filtered by a descending $\mathfrak{m}$-stable (or good) filtration of submodules $\mathbb{M} = \{\mathfrak{F}_p M\}_{p\geq 0}$. Define

$$gr_{\mathbb{M}}(M) = \bigoplus_{p\geq 0}(\mathfrak{F}_p M/\mathfrak{F}_{p+1} M)$$

the **associated graded module** to M with respect to the filtration $\mathbb{M}$. If $\mathbb{M} = \{\mathfrak{m}^p M\}$ is the $\mathfrak{m}$-adic filtration on M we denote $gr_{\mathbb{M}}(M)$ simply by M^g.

The main goal of this talk is to study properties of the module M, such as the linearity defect, the Koszulness and the regularity by means of the graded structure of $gr_{\mathbb{M}}(M)$ for a given $\mathfrak{m}$-stable filtration $\mathbb{M}$. The strength of our approach comes from a deep investigation of the interplay between a minimal free resolution of $gr_{\mathbb{M}}(M)$ as R^g-module and a free resolution of M as R-module coming from the theory of the standard bases.

We consider a minimal free resolution of M as finitely generated R-module

$$\mathbf{F} : \cdots \to F_{i+1} \xrightarrow{\phi_{i+1}} F_i \xrightarrow{\phi_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to 0$$

By definition, the i-th Betti number $\beta_i^R(M)$ of M is the rank of F_i, that is $\beta_i^R(M) = \dim_k \mathrm{Tor}_i^R(M, k)$. It is well known that, given a minimal graded free resolution $\mathbf{G}$ of

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$gr_{\mathbb{M}}(M)$ as R^g-module, one can build up a free resolution $\mathbf{F}$ of M (not necessarily minimal) equipped with a special filtration $\mathbb{F}$ on $\mathbf{F}$ such that $gr_{\mathbb{F}}(\mathbf{F}) = \mathbf{G}$ (see [3, 2.4]). As a consequence of this construction

$$\beta_i^R(M) \leq \beta_i^{R^g}(gr_{\mathbb{M}}(M))$$

and M is said of *homogeneous type* with respect to $\mathbb{M}$ if the equality holds for every $i \geq 0$. This is equivalent to say that the resolution $\mathbb{F}$ in the above construction is minimal.

If R is a standard graded k-algebra and M is a finitely generated graded R-module, then $\operatorname{Tor}_i^R(M,k)$ inherits the graded structure and the (i,j)-th graded Betti number $\beta_{ij}^R(M) = \dim_k \operatorname{Tor}_i^R(M,k)_j$ of M is the number of copies of $R(-j)$ that appear in F_i. We set

$$t_i^R(M) = \sup\{j : \beta_{ij}^R(M) \neq 0\}$$

where, by convention, $t_i^R(M) = -\infty$ if $F_i = 0$. By definition, $t_0^R(M)$ is the largest degree of a minimal generator of M. An important invariant associated to a minimal free resolution of M as R-module is the Castelnuovo-Mumford regularity

$$\operatorname{reg}_R(M) = \sup\{j - i : \beta_{ij}^R(M) \neq 0\} = \sup\{t_i^R(M) - i : i \in \mathbf{N}\}.$$

It is clear that $\operatorname{reg}_R(M)$ can be infinite. In the graded case we have a double opportunity: we can take advantage both of the graded R-structure of M and of the graded R-structure of $gr_{\mathbb{M}}(M)$. We present explicit bounds on the graded Betti numbers of M in terms of the degrees of a minimal system of generators of M and of the Betti numbers of $gr_{\mathbb{M}}(M)$. In particular we prove that if there exists a filtration $\mathbb{M}$ of M such that $\operatorname{reg}_R(gr_{\mathbb{M}}(M)) < \infty$, then $\operatorname{reg}_R(M) < \infty$. If M is of homogeneous type with respect to $\mathbb{M}$, then also the converse holds, see Corollary 2.2. Example 2.3 shows that the assumption is necessary.

If R is graded, the residue field k has a special behaviour. Avramov and Peeva in [1] proved that either k admits a linear resolution, that is $\operatorname{reg}_R(k) = 0$, or $\operatorname{reg}_R(k)$ is infinite. Following the classical definition given by Priddy, if k admits a linear resolution, we say that R is Koszul.

For local rings we say that R is Koszul if and only if R^g is Koszul. This concept is generalized by Herzog and Iyengar ([2]) for modules. Let $\mathbf{F}$ be the minimal free resolution of the $(R,\mathfrak{m})$- module M. The *standard filtration* $\mathfrak{F}$ of $\mathbf{F}$ is defined by subcomplexes $\{\mathfrak{F}_i\mathbf{F}\}$, where $(\mathfrak{F}^i\mathbf{F})_n = \mathfrak{m}^{i-n}\mathbf{F}_n$ for all $n \in \mathbb{Z}$. with $\mathfrak{m}^j = 0$ for $j \leq 0$. The associated graded complex with respect to this filtration is denoted by $\operatorname{lin}^R(\mathbf{F})$, and called the *linear part of* $\mathbf{F}$. The linearity defect of M is defined defined by

$$\operatorname{ld}_R(M) := \sup\{i \in \mathbb{Z} | \operatorname{H}_i(\operatorname{lin}^R(\mathbf{F}) \neq 0\}$$

M is called Koszul if $\operatorname{ld}_R(M) = 0$. In this talk we study the connection between the regularity and linearity defect of modules. We also give a partial answer to the following challenging question has been stated in [2, 1.14]

Question 1.1. If $\operatorname{ld}_R(k) < \infty$ does it follow that $\operatorname{ld}_R(k) = 0$?

This is an analogous of the statement proved by Avramov and Peeva for local rings replacing the concept of regularity with linearity defect.

In Section 2 and 3 we collect our main results.

2. Regularity of filtered graded modules

Let N be a graded R-module equipped with the filtration $\mathbb{N} = \{\mathfrak{F}_p N\}_{p\geq 0}$. For every non-zero homogeneous element $x \in N$ we have two integers attached to x. We say that x has degree i and we write $\deg(x) = i$ if $x \in N_i$ and we say that x has valuation $p = v_{\mathbb{N}}(x)$ if $x \in \mathfrak{F}_p N \setminus \mathfrak{F}_{p+1} N$. Also we denote by x^* the residue class of x in $\mathfrak{F}_p N/\mathfrak{F}_{p+1} N$.

Let $\{f_1, \cdots, f_s\}$ be a minimal homogeneous standard basis of N that is $\{f_1^*, \cdots, f_s^*\}$ forms a minimal generating set of $gr_{\mathbb{N}}(N))$. We set

$$\Delta_{\mathbb{N}}(N) = \{\deg(f_j) - v_{\mathbb{N}}(f_j) : 1 \leq j \leq s\}$$

and

$$v_{\mathbb{N}}(N) = \max \Delta_{\mathbb{N}}(N)$$
$$u_{\mathbb{N}}(N) = \min \Delta_{\mathbb{N}}(N)$$

Theorem 2.1. *With the above notation and assumptions, we have*

$$\operatorname{reg}_R(M) \leq \operatorname{reg}_R(gr_{\mathbb{M}}(M)) + v_{\mathbb{M}}(M).$$

Moreover if M is of homogeneous type with respect to $\mathbb{M}$, then

$$\operatorname{reg}_R(gr_{\mathbb{M}}(M)) + u_{\mathbb{M}}(M \leq \operatorname{reg}_R(M) \leq \operatorname{reg}_R(gr_{\mathbb{M}}(M)) + v_{\mathbb{M}}(M). \tag{1}$$

Corollary 2.2. *If $\operatorname{reg}_R(gr_{\mathbb{M}}(M)) < \infty$, then $\operatorname{reg}_R(M) < \infty$. Furthermore the converse holds, provided M is of homogeneous type with respect to $\mathbb{M}$.*

The assumption of being of homogeneous type cannot be deleted as the following example shows.

Example 2.3. Let k be a field and $R = k[X,Y]/(X^3)$. Let x, y be the residue class of X, Y in R and set $\mathfrak{m} = (x, y)$. Consider the module M whose minimal graded free resolution is

$$0 \to R(-3) \xrightarrow{\binom{x^2}{y^3}} R(-1) \bigoplus R(0) \to 0$$

The M is not of homogeneous type and $\operatorname{reg}_R(M) < \infty$, but

$$\operatorname{reg}_R(M^g) = \ +\infty.$$

3. Linearity defect and regularity

In this section we study connection between regularity and linearity defect of modules.

Proposition 3.1. *Let R be a standard graded algebra and let M be a finitely generated graded R-module. Let M generated in degrees $i_1, \cdots, i_s$. If M is Koszul. Then for each $n \geq 1$ we have*

$$\operatorname{Tor}_n^R(M,k)_j = 0 \quad for\ j \neq i_1 + n, \cdots, i_s + n.$$

Furthermore if for some $n \geq 1$ and $1 \leq r \leq s$, we have $\operatorname{Tor}_n^R(M,k)_{i_r+n} = 0$, then $\operatorname{Tor}_m^R(M,k)_{i_r+m} = 0$, for all $m \geq n$.

It is shown in [2] that if $\operatorname{ld}_R(M)$ is finite, then $\operatorname{reg}_R(M)$ is finite as well.
We can refine this result giving a different proof and a more precise information on the regularity.

Theorem 3.2. *Let R be a standard graded algebra and let M be a finitely generated graded R-module. If $\operatorname{ld}_R(M) = d < \infty$ then*

$$\operatorname{reg}_R(M) = \max\{t_i(M) - i : 0 \leq i \leq d\}.$$

In particular, if M is Koszul then $\operatorname{reg}_R(M) = t_0(M)$.

Theorem 3.3. *Let (R, m) be a local ring and let M be a Koszul module equipped with the filtration $\mathbb{M}$. If M is of homogeneous type with respect to $\mathbb{M}$, then*

$$\operatorname{reg}_{R^g}(gr_{\mathbb{M}}(M)) = t_0(gr_{\mathbb{M}}(M)).$$

Corollary 3.4. *Let $(R, \mathfrak{m})$ be a local ring and let M be a finitely generated R-module equipped with the filtration $\mathbb{M}$. Let $\operatorname{ld}_R(M) = d < \infty$ and assume that M is of homogeneous type with respect to $\mathbb{M}$, then*

$$\operatorname{reg}_{R^g}(gr_{\mathbb{M}}(M)) = \max\{t_i(gr_{\mathbb{M}}(M)) - i; 0 \leq i \leq d\}.$$

We are ready now to give a partial answer to Question 1.

Proposition 3.5. *Let $(R, \mathfrak{m}, k)$ be a local ring. If $\operatorname{ld}_R(k) < \infty$ and k is of homogeneous type, then $\operatorname{ld}_R(k) = 0$.*

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ON THE CHARACTERIZATION OF FINITE MOUFANG LOOPS BY THEIR NON-COMMUTING GRAPH

KARIM AHMADIDELIR

Abstract. The non–commuting graph Γ_G associated to a non–abelian group G, is a graph with vertex set $G \setminus Z(G)$ where distinct non–central elements x and y of G are joined by an edge if and only if $xy \neq yx$. The non–commuting graph of a non–abelian finite group has received some attention in existing literature.

Recently, many authors have studied the non–commuting graph associated to a non–abelian group. In particular Abdollahi-Akbari-Maimani (J. Algebra **298**:468–492, 2006), Moghaddamfar-Shi-Zhou-Zokayi (Siberian Math. J. **46**: 325–332, 2005) and Darafsheh (Discrete Appl. Math. **157**:833–837, 2009) put forward the following conjectures:

Conjecture 1. Let G and H be two non–abelian finite groups such that $\Gamma_G = \Gamma_H$. Then $|G| = |H|$.

Conjecture 2 (AAM's Conjecture). Let P be a finite non–abelian simple group and G be a group such that $\Gamma_G = \Gamma_P$. Then $G \cong P$.

Some authors have proved the first conjecture for some classes of groups (specially for all finite simple groups and non–abelian nilpotent groups with irregular isomorphic non–commuting graphs). However, Moghaddamfar in (Siberian Math.J.**47**: 911–914, 2006) has shown that the conjecture is not true in general by giving some counterexamples. On the other hand, Solomon and Woldar proved the second conjecture, in (J. Group Theory**16**:793–824, 2013).

In this talk, we will define the same concept for a finite non–commutative Moufang loop M and try to characterize some finite non–commutative Moufang loops with their non–commuting graph. Particularly, we obtain examples of finite non–associative Moufang loops and finite associative Moufang loops (groups) of the same order which have isomorphic non–commuting graphs. Also, we will obtain some results related to the non–commuting graph of a finite non–commutative Moufang loop. Finally, we give a conjecture stating that the above result is true for all finite simple Moufang loops.

1. Introduction

A set Q with one binary operation is a quasigroup if the equation $xy = z$ has a unique solution in Q whenever two of the three elements $x, y, z \in Q$ are specified. Loop is a quasigroup with a neutral element 1 satisfying $1x = x1 = x$ for every x.

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Moufang loops are loops in which any of the (equivalent) Moufang identities $((xy)x)z = x(y(xz)),\ x(y(zy)) = ((xy)z)y,\ (xy)(zx) = x((yz)x),\ (xy)(zx) = (x(yz))x$ holds.

Commutant (or *Moufang center* or *centrum*) of Q is defined by $\{x \in Q \mid xy = yx, \forall y \in Q\}$ and is denoted by $C(Q)$. *Center* of Q is defined by $\{x \in Q \mid [x, y] = [x, y, z] = [y, x, z] = 1\}$ and is denoted by $Z(Q)$. *Nucleus* of Q is denoted by $N(Q)$ and is the subset $\{x \in Q \mid x(yz) = (xy)z,\ y(xz) = (yx)z,\ y(zx) = (yz)x,\ \forall y, z \in Q\}$. A non–empty subset P of Q is called a *subloop* of Q if P is itself a loop under the binary operation of Q; in particular if this operation is associative on P, then it is called a *subgroup* of Q. A subloop $N \leq Q$ is called *normal* if $xN = Nx;\ x(yN) = (xy)N;\ N(xy) = (Nx)y$ for every $x, y \in Q$.

Now, $Z(Q) = C(Q) \cap N(Q)$, and $N(Q)$ and $Z(Q)$ are subgroups of Q, but in general, $C(Q)$ is not even a subloop. Of course, if Q be Moufang, then $C(Q)$ is a subloop of that (in fact, all of them, i.e. $N(Q)$, $Z(Q)$ and $C(Q)$, are normal in Q).

2. On the Classification of Moufang loops

To use in the next section, here we summerize the spectrum of Moufang loops: For which orders n is there a non–associative Moufang loop?

Chein and Rajah in [2] has proved a theorem and made an important corollary: a non–associative Moufang loop of order $2m$ exists if and only if a non–abelian group of order m exists. Hence, a non–associative Moufang loop of order 2^k exists if and only if $k > 3$, and for every odd $m > 1$ there is a non–associative Moufang loop of order $4m$.

Now, we know that, for any prime p, none of Moufang loops of order p, p^2, p^3 are non–associative; none of Moufang loops of order p^4 are non–associative unless $p = 2, 3$. Moufang loops of order pq, where p and q are distinct primes, must be associative. Also, Moufang loops of orders pqr, and p^2q, (p, q and r distinct odd primes) are all associative. By a theorem in [2], for odd primes $p < q$ a non–associative Moufang loop of order pq^3 exists if and only if $q \equiv 1\ (mod\ p)$. In [5] and [9], Goodaire, May, Raman, Nagy and Vojtěchovský classified all non–associative Moufang loops of order $3^4 = 81$ and 64. They proved that there are 4262 pairwise nonisomorphic non–associative Moufang loops of order 64, and there are five pairwise nonisomorphic non–associative Moufang loops of order 81, two of which are commutative. All five of these loops are isotopes of the two commutative ones. In [8], Nagy and Valsecchi have shown in that there are precisely four non–associative Moufang loops of order p^5 for every prime $p \geq 5$. Finally, Slattery and Zenisek classified all non–associative Moufang loops of order 243 (see [12]). So, the classification of non–associative Moufang loops of order p^4 and p^5 is now complete. Much more is known but the problem is open in general.

3. Preliminary Results

An elementary result in group theory says that, for a group G, if $G/Z(G)$ be cyclic, then G is abelian. We generalize it to all Moufang loops.

Proposition 3.1. *Let M be Moufang loop. If $M/Z(M)$ be cyclic, then M is commutative.*

Corollary 3.2. *For every non–commutative Moufang loop M,*

$$|M/Z(M)| = |M : Z(M)| \geq 4.$$

□

So, by the above corollary in every finite non–commutative Moufang loop M, the maximum number of elements of $Z(M)$ is $\frac{1}{4}$ of $|M|$. In the following result, we show that the same is true for $C(M)$, the commutant of M.

Proposition 3.3. *Let M be a finite non–commutative Moufang loop. Then*

$$|M : C(M)| \geq 4.$$

4. Some Properties of the non–commuting graph in finite Moufang loops

In this section, we state a few results on Γ_M, the non–commuting graph for a finite Moufang loop M. They can be compared with the similar results in finite groups (see, for example [1, 3, 7]).

Let Q be a loop. We define the non–commuting graph Γ_Q for Q, to be a graph with vertex set $V(\Gamma_Q) = Q \setminus C(Q)$, with two vertices x and y joined by an edge whenever the commutator of x and y is not the identity. Note that as in the groups, a finite loop Q is commutative if and only if $V(\Gamma_Q) = \emptyset$ and so Γ_Q is a null graph. If Q is an associative Moufang loop, then it is a group and $N(Q) = Q$, $Z(Q) = C(Q)$ and we have the same definition of non–commuting graph as in the groups. So, we used this definition and wrote some GAP codes with the aid of LOOPS and GRAPE packages ([4, 10, 13]). Our computation showed that for the non–associative finite Moufang loops of order n in the library (most cases where, $n \leq 64$ and $n \in \{81, 243\}$), we have connected non–commuting graph with diameter at most 3 and girth 3. But in general, we could only prove the connectedness of this graph with diameter at most 8!

In what follows we show that the non–commuting graph of every Moufang loop is always connected.

Proposition 4.1. *For any non–commutative finite Moufang loop M, Γ_M is connected with $2 \leq diam(\Gamma_M) \leq 8$ and $girth(\Gamma_M) = 3$.*

Also, we have the next result about partitioning the non–commuting graph of a finite Moufang loop.

Proposition 4.2. *For a finite non–commutative Moufang loop M, if $|M : Z(M)| = k$ and $|C(M) : Z(M)| = t$ then Γ_M is $(k - t)$–partite ($k \geq 4$). In particular, if $Z(M) = N(M) = C(M)$ and $|M : Z(M)| = k$, then Γ_M is $(k - 1)$–partite.*

5. Characterization of Moufang loops of small order by their non–commuting graph

In this section we want to characterize all Moufang loops up to order 63 and order $n = 81$ by their non–commuting graphs.

The following result is true in finite associative Moufang loops (groups, see lemma 1 in [7]) and we generalize it to all finite Moufang loops.

Lemma 5.1. *Let M be a finite Moufang loop with $|C(M)| = 1$ and order $p + 1$, p an odd prime. If L is any Moufang loop with $\Gamma_L \cong \Gamma_M$ then $|C(L)| = 1$ and $|L| = |M|$.*

Corollary 5.2. *Let M be a non–associative Moufang loop of order $4m$, where $m \geq 3$ is an odd positive integer. Let $|C(M)| = 1$ and $4m - 1$ be a prime number. Then for every Moufang loop L such that $\Gamma_L \cong \Gamma_M$, we have $|L| = |M|$. Particularly, if m is also a prime and L is non–associative, then $L \cong M$.*

From the above lemma or its corollary, we can characterize Moufang loops of orders $12, 20, 42$ and 44 by their non–commuting graphs.

Theorem 5.3. *Let M be a finite non–commutative non–associative Moufang loop. Let $|M| = 12, 20, 42$, or 44. If L is a Moufang loop with $\phi : \Gamma_L \cong \Gamma_M$, then $L \cong M$.*

Theorem 5.4. *Let M be a finite non–commutative non–associative Moufang loop of order 28. If L is a Moufang loop with $\Gamma_L \cong \Gamma_M$, then $L \cong M$.*

Theorem 5.5. *Let M be a finite non–commutative non–associative Moufang loop of 52. If L is a Moufang loop with $\Gamma_L \cong \Gamma_M$, then $L \cong M$.*

Denote by $n(\Gamma_M)$ the number of isomorphism classes of finite Moufang loops L satisfying $\Gamma_L \cong \Gamma_M$. A finite Moufang loop M is called k–recognizable if $n(\Gamma_M) = k < \infty$, otherwise M is called non–recognizable. Also a 1-recognizable Moufang loop is called a characterizable Moufang loop. Therefore, by the above theorems, all Moufang loops of orders $12, 20, 28, 42, 44, 52$ are 1–recognizable.

The following theorem shows that each of the five non–isomorphic non–associative Moufang loops of order 60 is 1–recognizable, i.e., characterizable by its non–commuting graph.

Theorem 5.6. *Every non–associative Moufang loops of order 60 are 1–recognizable.*

The following theorems show that non–associative Moufang loops of orders $54, 56$ are not characterizable by their non–commuting graphs. Infact each of them is 2–recognizable.

Theorem 5.7. *Let M be a finite non–commutative non–associative Moufang loop of order 54. Then every Moufang loop L with $\Gamma_L \cong \Gamma_M$, is $L \cong M(54,1) \cong M$ or $L \cong M(54,2) \cong M$. In addition, non–commuting graphs of $M(54,1)$ and $M(54,2)$ are isomorphic.*

Theorem 5.8. *Every non–commutative non–associative Moufang loop of order 56 is 2–recognizable. In addition, $\Gamma_{M(56,1)} \cong \Gamma_{M(56,3)}$ and $\Gamma_{M(56,2)} \cong \Gamma_{M(56,4)}$.*

The following theorem shows that non–commutative Moufang loops of orders 16 are not characterizable by their non–commuting graphs and they are 2– or 3–recognizable.

Theorem 5.9. *Every non–commutative non–associative Moufang loop of order 16 is 2–recognizable or 3–recognizable. In addition, $\Gamma_{M(16,1)} \cong \Gamma_{M(16,4)} \cong \Gamma_{M(16,5)}$ and $\Gamma_{M(16,2)} \cong \Gamma_{M(16,3)}$.*

The following theorems shows that non–commutative Moufang loops of orders 24, 36 and 40 are 1– or 2–recognizable.

Theorem 5.10. *Let $M_1 = M(24,1)$, $M_2 = M(24,2)$, $M_3 = M(24,3)$, $M_4 = M(24,4)$ and $M_5 = M(24,5)$. Then M_i, $i = 1,3,4,5$, are 2–recognizable and M_2 is 1–recognizable (characterizable). In addition, $\Gamma_{M_1} \cong \Gamma_{M_4}$ and $\Gamma_{M_3} \cong \Gamma_{M_5}$.*

Theorem 5.11. *Let $M_1 = M(36,1)$, $M_2 = M(36,2)$, $M_3 = M(36,3)$ and $M_4 = M(36,4)$. Then M_1 and M_4 are 1–recognizable (characterizable) and M_2 and M_3 are 2–recognizable. In addition, $\Gamma_{M_2} \cong \Gamma_{M_3}$.*

Theorem 5.12. *Let $M_1 = M(40,1)$, $M_2 = M(40,2)$, $M_3 = M(40,3)$, $M_4 = M(40,4)$ and $M_5 = M(40,5)$. Then M_3 is 1–recognizable (characterizable) and M_2, M_3, M_4 and M_5 are 2–recognizable. In addition, $\Gamma_{M_1} \cong \Gamma_{M_4}$ and $\Gamma_{M_2} \cong \Gamma_{M_5}$.*

The next theorem shows that non–commutative Moufang loops of order 81 are not characterizable by their non–commuting graphs. In fact, there are both non–isomorphic non–associative and non–isomorphic associative Moufang loops of order 81 with isomorphic non–commuting graphs.

Theorem 5.13. *If M is a non–commutative non–associative Moufang loop of order 81, and L is any Moufang loop such that $\Gamma_L \cong \Gamma_M$ then $|L| = |M|$. If L is also non–associative then M is 3-recognizable, i.e., $M \cong M(81,3) = L$, $M \cong M(81,4) = L$, $M \cong M(81,5) = L$. In fact, $\Gamma_{M(81,3)} \cong \Gamma_{M(81,4)} \cong \Gamma_{M(81,5)}$. But if L is associative then it is 6–recognizable. In addition, if we set $G_i = SmallGroup(81,i)$, then*

$$\Gamma_{G_3} \cong \Gamma_{G_4} \cong \Gamma_{G_6} \cong \Gamma_{G_{12}} \cong \Gamma_{G_{13}} \cong \Gamma_{G_{14}}.$$

In the next theorem we will show that the only characterizable non–associative Moufang loop of order 48 is $M(48,9) \cong M(S_4,2)$ and the rest of non–associative Moufang loops of this order are 2–, 3–, 4–, 5– or 6–recognizable. Also, some of them have isomorphic non–commuting graph with some groups of the same order.

Finally, in our last theorem we are going to show that none of the non–associative Moufang loops of order 32 is characterizable. If fact, non–associative Moufang loops of this order are 2–, 3–, 4–, 5–, 11– or 17–recognizable.

6. Characterization of Paige loops by their non–commuting graphs

In 1956, L. Paige in [11] constructed one Paige loop for every field $GF(q)$ (of course, he did not call them Paige loops). Thirty years later, M. Liebeck in [6] showed that there are no other Paige loops (non–associative finite simple Moufang loops). It is customary to denote the unique Paige loop constructed over $GF(q)$ by $M^*(q)$. An easy argument of Paige [11], shows that $M^*(q)$ has $q^3(q^4-1)$ elements when q is even, and $q^3(q^4-1)/2$ elements when q is odd. So, the smallest Paige loop is $M^*(2)$ of order 120.

In 2006, Abdollahi, Akbari and Maimani proposed a conjecture as follows [1]:

AAM's Conjecture. Let P be a finite non–abelian simple group and G be a group such that $\Gamma_G \cong \Gamma_P$. Then $G \cong P$.

Thereafter, this conjecture is verified for all sporadic simple groups, the alternating groups in some papers by the first author of [1] and some others, and finally Solomon and Woldar proved it in [14]. So, coming back to finite Moufang loops, it follows that every associative finite simple Moufang loop is characterizable by its non–commuting graph. Now, it is a natural question that what happens about non–associative finite simple Moufang loop? Can we characterize a Paige loop by its non–commuting graph? Formally, we propose it as a conjecture:

Conjecture: Let S be a finite non–commutative simple Moufang loop and L be a Moufang loop such that $\Gamma_L \cong \Gamma_S$. Then $L \cong S$.

So, to prove this conjecture it is enough to consider only "Paige loops".

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SOME PROPERTIES OF τ_M-LIFTING MODULES

TAYYEBEH AMOUZEGAR

Abstract. Let M be a right module over a ring R, τ_M a preradical on $\sigma[M]$. Let $N \in \sigma[M]$ be a τ_M-lifting module. Then there exist a semisimple submodule N_1 and a submodule N_2 of N such that $N = N_1 \oplus N_2$ and every nonzero submodule of N_2 contains a nonzero submodule H such that H is isomorphism to a submodule of $\tau_M(N)$.

1. Introduction

Throughout this talk R will denote an arbitrary associative ring with identity and all modules will be unitary right R-modules. Let $M \in$ Mod-R. By $\sigma[M]$ we mean the full subcategory of Mod-R whose objects are submodules of M-generated modules. For any module M, τ_M will denote a preradical in $\sigma[M]$. Recall that A is a *τ_M-cosmall submodule of B in N* if $B/A \subseteq \tau_M(N/A)$. According to [2], a module N is called *τ_M-lifting* if for every submodule K of N, there is a decomposition $K = A \oplus B$ such that A is a direct summand of N and $B \subseteq \tau_M(N)$. In this note, we study some properties of τ_M-lifting modules. Particularly, we prove the following main consequence:

(1) If τ_M is hereditary, then every direct summand of a τ_M-lifting module is τ_M-lifting.

(2) Let $N \in \sigma[M]$ be a τ_M-lifting module. If K is a fully invariant submodule of N, then N/K is τ_M-lifting.

(3) Let $N \in \sigma[M]$ be a τ_M-lifting module. Then there exist a semisimple submodule N_1 and a submodule N_2 of N such that $N = N_1 \oplus N_2$ and every nonzero submodule of N_2 contains a nonzero submodule H such that H is isomorphism to a submodule of $\tau_M(N)$.

2. Main Results

A module N is called *τ_M-lifting* if for every submodule K of N, there is a decomposition $K = A \oplus B$ such that A is a direct summand of N and $B \subseteq \tau_M(N)$.

Lemma 2.1. *Let $N \in \sigma[M]$ be a τ_M-lifting module and K a submodule of N. Then either K contains a nonzero submodule H such that $H \subseteq \tau_M(N)$ or K is a semisimple direct summand of N.*

Theorem 2.2. *Let $N \in \sigma[M]$ be a τ_M-lifting module. Then there exist a semisimple submodule N_1 and a submodule N_2 of N such that $N = N_1 \oplus N_2$ and every nonzero*

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submodule of N_2 contains a nonzero submodule H such that H is isomorphism to a submodule of $\tau_M(N)$.

A preradical τ_M is called a *hereditary preradical* if for any submodule K of $N \in \sigma[M]$, $\tau_M(K) = \tau_M(N) \cap K$. But it is well known that if K is a direct summand of N, then $\tau_M(K) = \tau_M(N) \cap K$.

Proposition 2.3. *If τ_M is a hereditary preradical, then every submodule of a τ_M-lifting module is τ_M-lifting.*

Corollary 2.4. *If τ_M is hereditary, then every direct summand of a τ_M-lifting module is τ_M-lifting.*

Theorem 2.5. *Let $N_1, N_2 \in \sigma[M]$ be two τ_M-lifting modules such that N_i is N_j-projective $(i, j = 1, 2)$. Then $N = N_1 \oplus N_2$ is τ_M-lifting.*

Corollary 2.6. *Let $N_1, N_2 \in \sigma[M]$ be two projective τ_M-lifting modules. Then $N = N_1 \oplus N_2$ is τ_M-lifting.*

Recall that a submodule K of M is called *fully invariant* (denoted by $K \trianglelefteq M$) if $\lambda(K) \subseteq K$ for all $\lambda \in End_R(M)$. A module $N \in \sigma[M]$ is called a *duo module* provided every submodule of N is fully invariant.

Theorem 2.7. *Let $N \in \sigma[M]$ and $N = N_1 \oplus N_2$ be a duo module such that N_1 and N_2 are two τ_M-lifting modules. Then N is τ_M-lifting.*

Proposition 2.8. *Let τ_M be a hereditary preradical. Then the following conditions are equivalent:*
(i) *Every module is τ_M-lifting.*
(ii) *Every injective module is τ_M-lifting.*

Proposition 2.9. *Let $N \in \sigma[M]$ be a τ_M-lifting module. Then:*
(i) *If K is a fully invariant submodule of N, then N/K is τ_M-lifting.*
(ii) *If K is a submodule of N such that the sum of K with any nonzero direct summand of N is a direct summand of N, then N/K is τ_M-lifting.*

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CHARACTERIZATION OF SOME SIMPLE GROUPS BY THE NUMBER OF SYLOW SUBGROUPS

B. ASADIAN†

Abstract. In this talk, we study the characterization of some simple groups by the number of their Sylow subgroups in the class of finite centerless groups.

This is a joint work with N. Ahanjideh‡.

1. Introduction

For a finite group G and the prime number p, let $n_p(G)$ denote the number of p-Sylow subgroups of G and $\pi(G)$ be the set of prime divisors of $|G|$. For the natural number n, a finite group S is named a simple K_n-group, when S is a simple group with $|\pi(S)| = n$. A finite group G is said to be characterizable by the number of its Sylow subgroups in the class of finite centerless groups, if G is uniquely (up to isomorphism) determined by the number of its Sylow subgroups in the class of finite centerless groups. In [4], the authors showed that some finite simple groups are characterizable by the number of their Sylow subgroups in the class of finite centerless groups. In this paper, S is a simple K_4-group $L_2(q)$, where $q = p^m$ and $p \in \{2, 3\}$. The goal of this paper is to prove that:

Main Theorem If G is a finite centerless group and for every prime $p \in \pi(G)$, $n_p(G) = n_p(S)$, then $G \cong S$.

2. Main Results

In the following, we bring some lemmas which are useful in the proof of our main result:

Lemma 2.1. (i) [3] *If G is a simple K_3-group, then G is isomorphic to one of the following groups: A_5, A_6, $PSL(2,7)$, $PSL(2,8)$, $PSL(2,17)$, $PSL(3,3)$, $PSU(3,3)$ or $PSU(4,2)$.*

(ii) [5] *let G be a finite group and M be a normal subgroup of G. Then for every prime p, $n_p(M)n_p(G/M)$ divides $n_p(G)$.*

(iii) *let G be a simple K_4-group. Then G is isomorphic to one of the following groups: (1) A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2, $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$,*

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$O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^3D_4(2)$, ${}^2F_4(2)'$;

(2) $L_2(r)$, *where* r *is a prime and satisfies* $r^2-1=2^a.3^b.v^c$ *with* $a,b,c\geq 1$ *and a prime* $v>3$;

(3) $L_2(2^m)$, *where* $m\geq 5$ *is a prime number satisfies* $2^m-1=u$ *and* $2^m+1=3t^b$, *where* u *and* t *are primes,* $t>3$ *and* $b\geq 1$;

(4) $L_2(3^m)$, *where* $m\geq 3$ *is a prime number satisfies* $3^m+1=4t$, $3^m-1=2u^c$ *or* $3^m+1=4t^b$, $3^m-1=2u$, *where* u *and* t *are odd primes, and* $b,c\geq 1$.

(iv) [2] *Let* G *be a finite solvable group and* $|G|=m.n$, *where* $m=p_1^{\alpha_1}...p_r^{\alpha_r}$ *and* $(m,n)=1$. *Let* $\pi=\{p_1,...,p_r\}$ *and* h_m *be the number of* π*-Hall subgroups of* G. *Then* $h_m=q_1^{\beta_1}...q_s^{\beta_s}$, *where* $q_i^{\beta_i}\equiv 1 \pmod{p_j}$, *for some* p_j.

3. Proof of the Main Theorem

In this section we are equipped to prove the main theorem in five steps. According to our assumption G is a finite centerless group, S is a simple K_4-group $L_2(q)$, where $q=p^m$ and $p\in\{2,3\}$ and for every $p\in\pi(G)$, $n_p(G)=n_p(S)$.

Step (i) $\pi(G)=\pi(S)$.

Proof. On the contrary, suppose that there exists $p\in\pi(G)\setminus\pi(S)$, so our assumption on the number of Sylow subgroups of G shows that $p\in\pi(Z(G))$, which is a contradiction. □

Step (ii) G is a non-solvable group.

Proof. On the contrary, we assume that G is a solvable group.

If $S=L_2(2^m)$, then since $n_u(G)=n_u(L_2(2^m))=2^{m-1}.3.t^b$, we deduce that $u \mid 2^{m-1}-1$, $u\mid 3-1$ or $u\mid t^b-1$, by Lemma 2.1(iv)(1). We arrive at a contradiction. Also, if $S=L_2(3^m)$, then since $n_t(G)=n_t(L_2(3^m))=3^{m-1}.u^c$, where $u^c=(3^m-1)/2$, we deduce that by Lemma 2.1(iv)(1), $t\mid 3^{m-1}-1$ or $t\mid u^c-1$, which shows that $t\mid u^c-1=3(3^{m-1}-1)/2$. This forces t to divide $3^{m-1}-1$, which is impossible. □

Let $1=N_0\trianglelefteq N_1\trianglelefteq ...\trianglelefteq N_{n-1}\trianglelefteq N_n=G$ be a chief series of G. Using step(ii), there exists $1\leq i\leq n$ such that N_i/N_{i-1} is not solvable. Put $N:=N_{i-1}$ and $H:=N_i$. Since H/N is a normal minimal subgroup of G/N, $H/N\cong P_1\times ...\times P_t$, where for every $1\leq i\leq w$, P_i is a simple group and $P_i\cong P_j$, for every $1\leq i,j\leq t$. Fix j such that $1\leq j\leq t$. According to our assumption, S is a K_4-group. It is evident that P_j is neither a K_1-group nor a K_2-group, so we examine P_j in the following steps as a K_3-group or a K_4-group.

Step (iii) For every $1\leq j\leq t$, P_j is not a simple K_3-group.

Proof. If $S=L_2(2^m)$, then by Lemma 2.1(i) and, since u is the maximal prime divisor of $|G|$ and t is the maximal prime divisor of $|G|/|G|_u$, we deduce that $u\in\{5,7,13,17\}$ or $u\notin\pi(H/N)$ and $t\in\{5,7,13,17\}$. Also, if $S=L_2(3^m)$, then since $\pi(P_j)\subset\pi(G)=\{2,3,t,u\}$ and $|\pi(P_j)|=3$, we deduce that either $t\in\pi(P_j)$ or $u\in\pi(P_j)$. In every case, the assumption in Lemma 2.1(iii) or comparing the maximal prime divisor of the groups mentioned in Lemma 2.1(i) with t and u leads us to get a contradiction. □

Step (iv) Let P_j be a simple K_4-group mentioned in above, then $P_j \cong S$.

Proof. It is sufficient to compare the maximal prime divisor of order of groups mentioned in Lemma 2.1(iii) with the maximal prime divisor of $|S|$. Applying this manner or Lemma 2.1(ii) follows the result. □

From Step(iv) and Lemma 2.1(ii), we deduce $H/N \cong S$. We define $\overline{G} := G/N$ and $\overline{H} := H/N$. By using N-C Theorem, we have:

$$S \cong \overline{H} \cong \frac{\overline{H}C_{\overline{G}}(\overline{H})}{C_{\overline{G}}(\overline{H})} \leq \frac{\overline{G}}{C_{\overline{G}}(\overline{H})} = \frac{N_{\overline{G}}(\overline{H})}{C_{\overline{G}}(\overline{H})} \lesssim \mathrm{Aut}(\overline{H}) \cong \mathrm{Aut}(S).$$

Let $C_{\bar{G}}(\bar{H}) = K/N$. Then $K \trianglelefteq G$ and $G/K \cong \overline{G}/C_{\overline{G}}(\overline{H})$ and hence, $S \trianglelefteq G/K \lesssim \mathrm{Aut}(S)$. Our proof starts with the assumption that $G/K \cong S$. We claim that $K = 1$. It is sufficient to show $Q = 1$, for every q-Sylow subgroup Q of K. By Lemma 2.1(ii), it is easily seen that $n_p(K) = 1$, for every prime $p \in \pi(G)$. Let Q be a q-Sylow subgroup of K. Then since K is nilpotent, Q is normal in G. Now let T be a q-Sylow subgroup of G. We can check that $C_Q(T) \leq Z(G) = 1$ and hence, since q is an arbitrary element of $\pi(G)$, $K = 1$.

Step (v) $G \cong S$.

Proof. If $S = L_2(2^m)$, then since $K = 1$, $G \cong L_2(2^m) : \mathbb{Z}_n$, where $n \mid m$. But $u = 2^m - 1$ and $t^b = (2^m + 1)/3$. Applying Fermat's little theorem and $\pi(G) = \pi(S)$ imply that $n = 2^\alpha 3^\beta \mid m$, where $\alpha,\ \beta \in \mathbb{N} \cup \{0\}$. Also, "$2^m - 1 = u$" forces m to be a prime number and thus, $m = 2$, $m = 3$ or $n = 1$ and hence, our assumption forces $n = 1$, as wanted. If $S = L_2(3^m)$, then either $G \cong PGL_2(3^m) : \mathbb{Z}_n$ or $G \cong L_2(3^m) : \mathbb{Z}_n$, where $n \mid m$. If $G \cong PGL_2(3^m) : \mathbb{Z}_n$, then by Lemma 2.1(ii) $n_2(PGL_2(3^m))$ divides $n_2(G) = n_2(L_2(3^m))$ and [1, Lemma 3] shows that $|PGL_2(3^m)|/|PGL_2(3^m)|_2$ divides $|L_2(3^m)|/3.|L_2(3^m)|_2 = |PGL_2(3^m)|/3|PGL_2(3^m)|_2$, which is impossible. Thus $G \cong L_2(3^m) : \mathbb{Z}_n$. Then since $n \mid m$ and m is a prime number, we get either $n = 1$ or $n = m$. We claim that $n = 1$. If not, $n = m \in \pi(G) = \pi(L_2(3^m)) = \{2, 3, u, t\}$. Applying Fermat's little theorem shows that $t, u \nmid m$, consequently, $m \neq u$ and $m \neq 3$. Thus $n = m \in \{2, 3\}$ which in every case we arrive at a contradiction. This shows that $n = 1$ and hence, $G \cong L_2(3^m)$. □

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A NOTE ON THE AUSLANDER-REITEN CONJECTURE

ABDOLNASER BAHLEKEH

Abstract. The Auslander-Reiten conjecture claims that over an Artin algebra Λ, if M is a finitely generated Λ-module such that $\mathrm{Ext}_{\Lambda}^{i>0}(M, M \oplus \Lambda) = 0$, then M is projective. This conjecture which is rooted in Nakayama's 1958 conjecture on algebras of infinite dominant dimension, makes sense for arbitrary rings. Recently Araya proved that if the Auslander-Reiten conjecture holds in codimension one for a commutative Gorenstein ring R, then it holds for R. In this note we extend Araya's result to R-algebras Λ in which $\mathrm{Hom}_R(\Lambda, R)$ has finite projective dimension over Λ^{op}. As an application, we include some examples of algebras which satisfy the Auslander-Reiten conjecture.

This is joint work with A. Mahin Fallah and Sh. Salarian

1. Introduction

The generalized Nakayama conjecture of Auslander and Reiten predicts that, for an Artin algebra Λ, every indecomposable injective Λ-module appears as a direct summand in the minimal injective resolution of Λ; see [2]. They showed that this conjecture holds for all Artin algebras if and only if the following conjecture holds for all Artin algebras.

Conjecture . Let Λ be an Artin algebra and M a finitely generated Λ-module. If $\mathrm{Ext}\,_{\Lambda}^{i>0}(M, M \oplus \Lambda) = 0$, then M is projective.

This long-standing conjecture which is known as *the Auslander-Reiten conjecture*, is proved affirmatively for several classes of algebras, including algebras of finite representation type [2] and symmetric Artin algebras with radical cube zero [4]. The Auslander-Reiten conjecture is also called *Gorenstein projective conjecture*, whenever the module M (in the formulation of the conjecture) is assumed in addition to be Gorenstein projective over Λ; see [6]. It is easy to see that if Λ is Gorenstein, i.e. Λ has finite self-injective dimension on both sides, then the Auslander-Reiten conjecture and Gorenstein projective conjecture coincide. It is known that Gorenstein projective conjecture holds for algebras of finite Cohen-Macaulay type, however the validity of Auslander-Reiten conjecture for this class of algebras is unknown. The Auslander-Reiten conjecture has recently received considerable attention in commutative rings. In particular, Auslander, Ding and Solberg in [3], studied the following condition on a commutative noetherian ring which is not necessarily an Artin algebra.

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(ARC) Let R be a commutative noetherian ring and M a finitely generated R-module. If $\operatorname{Ext}_R^{i>0}(M, M \oplus R) = 0$, then M is projective.

Some of the well-known examples of commutative noetherian local rings that satisfy (ARC), are Gorenstein local rings of codimension at most four [7] and Gorenstein rings that are complete intersections in codimension one [5]. Recently, Araya [1] proved that the validity of (ARC) for the class of commutative Gorenstein rings depends on its validity for such rings of dimension at most one. In view of the importance of the Auslander-Reiten conjecture, it is highly desirable to have as much as possible information about classes of algebras satisfying the conjecture. This note, which is based on a joint work with A. Mahin Fallah and Sh. Salarian, extends Araya's result to R-algebras Λ in which the $\Lambda^{\rm op}$-module $\operatorname{Hom}_R(\Lambda, R)$ has finite projective dimension, where R is a commutative Gorenstein ring. Indeed, we deduce Aray's result from a more general context.

Throughout this talk, R is a commutative Gorenstein local ring and Λ is an R-algebra which is a finitely generated (maximal) Cohen-Macaulay R-module. By a module, we always mean a finitely generated left module.

2. Results

The following is a main result of my talk.

Theorem 2.1. *Let R be a ring with $\dim R \geq 2$ and let Λ be an R-algebra in which the $\Lambda^{\rm op}$-module $Hom_R(\Lambda, R)$ has finite projective dimension. Assume that M is a Gorenstein projective Λ-module which is locally projective on the punctured spectrum of R. If $Ext_\Lambda^{i>0}(M, M) = 0$, then M is a projective Λ-module.*

Recall that a Λ-module M is locally projective on the punctured spectrum of R, provided $M_\mathfrak{p}$ is projective over $\Lambda_\mathfrak{p}$, for all nonmaximal prime ideals $\mathfrak{p}$ of R.

The above theorem leads us to extend Araya's result to a more general situation. Indeed, we have the sequel result.

Theorem 2.2. *Let R be a ring and Λ an R-algebra in which the $\Lambda^{\rm op}$-module $Hom_R(\Lambda, R)$ has finite projective dimension. Let M be a Gorenstein projective Λ-module such that $Ext_\Lambda^{i>0}(M, M) = 0$. If $M_\mathfrak{p}$ is a projective $\Lambda_\mathfrak{p}$-module, for all prime ideals $\mathfrak{p}$ of height at most one, then M is projective over Λ.*

Corollary 2.3. *Let R be a ring with $dim\, R = d \geq 2$ and let Λ be an isolated singularity in the sense that $l.\mathrm{gldim}\Lambda_\mathfrak{p}$ is finite, for all nonmaximal prime ideals $\mathfrak{p}$ of R. Then Λ satisfies Gorenstein projective conjecture.*

The result below characterizes R-algebras Λ in which the $\Lambda^{\rm op}$-module $\operatorname{Hom}_R(\Lambda, R)$ has finite projective dimension. Consequently, there are plenty examples of algebras satisfying the assumption. We say that Λ is left Gorenstein, provided it has finite left injective dimension as a left Λ-module.

Proposition 2.4. *Let Λ be an R-algebra. Then Λ is left Gorenstein if and only if $proj.dim_{\Lambda^{\rm op}}(Hom_R(\Lambda, R)) < \infty$.*

As applications of the above theorems, we include some examples. A Gorenstein local ring R is said to satisfy *Serre's condition* (R_1), provided $R_\mathfrak{p}$ is regular, for all prime ideals $\mathfrak{p}$ of R with ht$\mathfrak{p} \leq 1$. One should note that since R is assumed to be Gorenstein, the same is true for $R\Gamma$ and RQ, whenever Γ is a finite group and Q is a finite acyclic quiver.

Example 2.5. (*i*) Let R be a Gorenstein ring which satisfies Serre's condition (R_1) and let Q be a finite acyclic quiver. Then the path algebra RQ satisfies (ARC).

(*ii*) Let Γ be a finite group. Assume that M is an $R\Gamma$-module in which $\operatorname{Ext}_{R\Gamma}^{i>0}(M, M \oplus R\Gamma) = 0$. Suppose that $M_{\mathfrak{p}}$ is a projective $(R\Gamma)_{\mathfrak{p}}$-module, for all prime ideals $\mathfrak{p}$ of R of height at most one. Then M is indeed projective over $R\Gamma$.

(*iii*) Let Q be a finite acyclic quiver and let M be an RQ-module in which $\operatorname{Ext}_{RQ}^{i>0}(M, M \oplus RQ) = 0$. Suppose that $M_{\mathfrak{p}}$ is a projective $(RQ)_{\mathfrak{p}}$-module, for all prime ideals $\mathfrak{p}$ of R of height at most one. Then M is indeed projective over RQ.

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DISCRETE LOGARITHM PROBLEM

MOJTABA BAHRAMIAN

Abstract. In this talk we review the discrete logarithm problem, and we reduce the discrete logarithm problem on some elliptic curves and the generalized Jacobian of elliptic curves to the discrete logarithm in the multiplicative group of $\mathbb{F}_q$.

1. Introduction

1.1. **Elliptic Curves.** An elliptic curve E over a field K is a smooth algebraic curve defined by an equation of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{1}$$

with coefficients in $\mathbb{K}$, and a specified point $\mathcal{O}$. The point $\mathcal{O}$ is actually the "point at infinity" in the projective plane. If char K is not equal to 2 and 3, then the equation (1) can be written as the form $y^2 = x^3 + Ax + B$. If we want to consider points with coordinates in K, we write $E(K)$. By definition, this set always contains the point $\mathcal{O}$ and therefore

$$E(K) = \{(x, y) \in K \times K : y^2 = x^3 + Ax + B\} \cup \{\mathcal{O}\}.$$

An elliptic curve is also naturally an abelian group. The group law is constructed geometrically. If E is an elliptic curve over the finite field $\mathbb{F}_q$, then the group $E(\mathbb{F}_q)$ is finite and $|q + 1 - \sharp E(\mathbb{F}_q)| \leq 2\sqrt{q}$. $a = q + 1 - \sharp E(\mathbb{F}_q)$ is called the trace of E.

1.2. **Discrete Logarithm Problem.** Fix a group G and an element $g \in G$. The Discrete Logarithm Problem (DLP) for G is:

Given an element $h \in \langle g \rangle$, find an integer m satisfying $h = g^m$.

The smallest integer m satisfying $h = g^m$ is called the logarithm of h with respect to g. The Discrete Logarithm Problem is used as the underlying hard problem in many cryptographic constructions. For some groups, DLP is very easy and for some is difficult. In general the best known algorithm to solve the Elliptic Curve Discrete Logarithm Problem (ECDLP) is exponential. So the group of elliptic curves is suitable for cryptography. But for certain special classes of curves, there are faster methods. For the elliptic curves of trace 1 and 2, there exist algorithms to solve ECDLP in time O(log q).

In [5], Smart describes an elementary technique which leads to a linear algorithm for solving the discrete logarithm problem on elliptic curves of trace one.

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Let E be an elliptic curve of trace two over the finite field $\mathbb{F}_q$. Frey, Muller and Ruck used The Tate pairing to reduce the DLP on $E(\mathbb{F}_q)$ to the DLP in the multiplicative group of $\mathbb{F}_q$ [3].

Also, Daghigh and Bahramian used the generalized jacobian of elliptic curves and showed that how the DLP on the elliptic curves of trace 2 can be reduce to the DLP over $\mathbb{F}_q^*$ [1].

2. Generalized Jacobian of Elliptic Curves

Let E be an elliptic curve over a field K. Also let $Div(E)$ be the free abelian group of all divisors of E, $Div^0(E)$ be the subgroup of $Div(E)$ of divisors of degree zero, and Princ(E) be the group of all principal divisors. The quotient group $Div^0(E)/Princ(E)$ is called the Jacobian of E and is denoted by $J(E)$. It is known that $J(E)$ is isomorphic to the group E ([4] Proposition III.3.4).

Now let $\mathfrak{m} = \sum_{P \in E} m_P(P)$ be an effective divisor and let D and D' be two divisors with supports disjoint from the support of $\mathfrak{m}$. Divisors D and D' are called $\mathfrak{m}$-equivalent, if there is a function $f \in K(E)$ such that $div(f) = D - D'$ and $ord_P(1-f) \geq m_P$ for each P in the support of $\mathfrak{m}$.

Now let

$$\begin{aligned} Div_{\mathfrak{m}}(E) &= \{D \in Div(E) : supp(D) \cap supp(\mathfrak{m}) = \emptyset\}, \\ Div^0_{\mathfrak{m}}(E) &= Div_{\mathfrak{m}}(E) \cap Div^0(E), \\ Princ_{\mathfrak{m}}(E) &= Div_{\mathfrak{m}}(E) \cap Princ(E). \end{aligned}$$

The quotient group $Div^0_{\mathfrak{m}}(E)/Princ_{\mathfrak{m}}(E)$ is called the Generalized Jacobian of E with respect to $\mathfrak{m}$ and is denoted by $J_{\mathfrak{m}}(E)$.

Now let E be an elliptic curve defined over the finite field $\mathbb{F}_q$ and suppose that M and N are distinct nonzero points of $E(\mathbb{F}_q)$ and let $\mathfrak{m} = (M) + (N)$. Dechene [2] showed that there exists the short exact sequence

$$1 \to \mathbb{F}_q^* \to J_{\mathfrak{m}}(E) \to E \to 0.$$

This implies that $J_{\mathfrak{m}}$ will be an extension of the elliptic curve E by the multiplicative group $\mathbb{F}_q^*$. Hence each element of $J_{\mathfrak{m}}(E)$ can be represented as a pair (k, P), where $k \in \mathbb{F}_q^*$ and $P \in E$. Moreover, If (k_1, P_1) and (k_2, P_2) are elements of $J_{\mathfrak{m}}$ fulfilling $P_1, P_2, \pm(P_1 + P_2) \notin \{M, N\}$, then

$$(k_1, P_1) + (k_2, P_2) = (k_1 k_2 c_{\mathfrak{m}}(P_1, P_2), P_1 + P_2),$$

where $c_{\mathfrak{m}}(P_1, P_2) = \frac{g(M)}{g(N)}$, and $div(g) = (P_1) + (P_2) - (P_1 + P_2) - (\mathcal{O})$.

3. Solving DLP in the Elliptic Curves of Trace 2

Let E be an elliptic curve over the finite field $\mathbb{F}_q$ and $\mathfrak{m} = (M) + (N)$ be a divisor. Also let

$$\alpha_t(P) = c_{\mathfrak{m}}(P, P) c_{\mathfrak{m}}(P, 2P) \cdots c_{\mathfrak{m}}(P, (t-1)P)$$

be the the first component of $t(1, P)$ where $(1, P) \in J_{\mathfrak{m}}$. Daghigh and Bahramian in [1] proved that if $n | q - 1$ and $P \in E$ be an element of order n, then

$$\alpha_n(kP)^{q-1} = (\alpha_n(P)^{q-1})^k$$

for all integer k. This equation reduces the DLP on $E(\mathbb{F}_q)$ to the DLP in the multiplicative group of $\mathbb{F}_q$.

4. DLP in Generalized Jacobian of Elliptic Curves

In the generalized jacobian $J_{\mathfrak{m}}$ let (a, P) and (b, Q) be the representation of two elements of $J_{\mathfrak{m}}$ such that $(b, Q) = k(a, P)$. Also let $ord(P) = n$. In this case $b = a^k \alpha_k(P)$ and therefore

$$b^n \alpha_n(Q) = (a^n \alpha_n(P))^k.$$

This equation gives a discrete logarithm problem in $\mathbb{F}_q^*$.

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WELL-BEHAVIOR OF STRONGLY S-DENSE INJECTIVITY AND ESSENTIALITY OF ACTS

H. BARZEGAR

Abstract. For a class $\mathcal{M}$ of monomorphisms of a category, mathematicians usually use different types of essentiality. Essentiality is an important notion closely related to injectivity. Banaschewski defines and gives sufficient conditions on a category $\mathcal{A}$ and a subclass $\mathcal{M}$ of its monomorphisms under which $\mathcal{M}$-injectivity *well-behaves* with respect to the notions such as $\mathcal{M}$-absolute retract and $\mathcal{M}$-essentialness.

In this talk, $\mathcal{A}$ is taken to be the category of acts over a semigroup S and $\mathcal{M}_{sd}$ to be the class of strongly s-dense monomorphisms. We study essentiality with respect to strongly s-dense monomorphisms of acts. Depending on a class $\mathcal{M}$ of morphisms of a category $\mathcal{A}$, three different types of essentialness are considered by H. Barzegar, M.M. Ebrahimi, and M. Mahmoudi in (Applied Categorical Structure, 2008).

1. Introduction

An important notion related to injectivity with respect to monomorphisms or any other class $\mathcal{M}$ of morphisms in a category $\mathcal{A}$ is essentiality. In fact, injectivity is characterized and injective hulls are defined using essentiality (see, for example, [1] and [3]). Recall that for a subclass $\mathcal{M}$ of the class $\mathcal{M}ono$ of monomorphisms of a category $\mathcal{A}$ and $M \stackrel{m}{\to} X \in \mathcal{M}$, one usually uses one of the following definitions to say that m is *essential*:

$\mathcal{M}_{e1}$: $M \stackrel{m}{\rightarrowtail} X \stackrel{f}{\to} Y \in \mathcal{M} \Rightarrow f \in \mathcal{M}$.

$\mathcal{M}_{e2}$: $M \stackrel{m}{\rightarrowtail} X \stackrel{f}{\to} Y \in \mathcal{M}ono \Rightarrow f \in \mathcal{M}ono$.

$\mathcal{M}_{e3}$: $M \stackrel{m}{\rightarrowtail} X \stackrel{f}{\to} Y \in \mathcal{M} \Rightarrow f \in \mathcal{M}ono$.

Clearly $\mathcal{M}_{e3}$ is weaker than the other two and if $\mathcal{M}$ is taken to be the class $\mathcal{M}ono$ of all monomorphisms (in which case m is said to be an *essential monomorphism*), all the above three conditions are equivalent. Definition $\mathcal{M}_{e1}$ is usually used for an arbitrary class $\mathcal{M}$ of morphisms of an arbitrary category $\mathcal{A}$ (see [1]). The second is the one which is used in Universal Algebra, and the third ones have been used when $\mathcal{M}$ is a special class of monomorphisms, in particular pure monomorphisms in an equational class of algebras.

Recall also that, there are three propositions first given by Banaschewski, mainly about the relation between injectivity, retractness, essentialness, and injective hulls, all with respect to a class $\mathcal{M}$ of morphisms of a category, which if they hold true one says that $\mathcal{M}$-injectivity "Well Behaves" (see, for example, [1]).

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In this paper, we take $\mathcal{A} = \textbf{Act-S}$ to be the category of right acts over a semigroup S and $\mathcal{M} = \mathcal{M}_{sd}$ to be the class of strongly s-dense monomorphisms of right S-acts, which studied in [2], and then we study the above notions of essentiality with respect to this class.

2. Essentiality with respect to strongly s-dense monomorphisms

Now that we have introduced the class $\mathcal{M}_{sd}$ of st-s-dense monomorphisms, which studied in [2], we begin the study of essentiality with respect to this class.

Definition 2.1. An S-act A is *strongly-s-dense(or simply st-s-dense)* subact of B, if for every $b \in B, bS \subseteq A$ and for every finite subset T of S there is an element $a_T \in A$ such that $a_T t = bt (t \in T)$. A homomorphism $f : A \to B$ is said to be st-s-dense if $f(A)$ is an st-s-dense subact of B.

Theorem 2.2. *For an st-s-dense monomorphism $f : A \to B$, the following are equivalent:*

(i) *Any homomorphism $g : B \to C$ for which gf is a monomorphism is itself a monomorphism.*

(ii) *Any homomorphism $g : B \to C$ for which gf is an st-s-dense monomorphism is itself an st-s-dense monomorphism.*

(iii) *Any homomorphism $g : B \to C$ for which gf is an st-s-dense monomorphism is a monomorphism.*

Definition 2.3. We call an st-s-dense monomorphism satisfying any one of the equivalent conditions of the above theorem a *st-s-dense essential extension* or $\mathcal{M}_{sd}$-*essential*, for short.

Lemma 2.4. *For an st-s-dense monomorphism $f : A \to B$, the following are equivalent:*

(i) *f is an st-s-dense essential extension.*

(ii) *For every epimorphism $g : B \to C$ such that gf is a monomorphism, g itself is a monomorphism.*

(iii) *For every congruence ρ on B such that for the canonical epimorphism $\pi : B \to B/\rho$, πf is a monomorphism , we get $\rho = \Delta$.*

(iv) *For every monogenic congruence ρ on B such that for the canonical epimorphism $\pi : B \to B/\rho$, πf is a monomorphism, we get $\rho = \Delta$.*

The following lemma shows that the class $\mathcal{M}_{sd}$- essentials is composition closed, also has left and right cancellable property.

Lemma 2.5. *The monomorphisms $f : A \to B$ and $g : B \to C$ are st-s-dense essential extensions if and only if gf is an st-s-dense essential extension.*

Proposition 2.6. **Act-S** *fulfills Banaschewski's $\mathcal{M}_{sd}$-condition, that is, for every $\mathcal{M}_{sd}$-monomorphism $f : A \to B$ there exists a homomorphism $g : B \to C$ such that gf is an $\mathcal{M}_{sd}$-essential monomorphism.*

Definition 2.7. Let A be an S-act. Then

(i) An $\mathcal{M}_{sd}$-essential extension f is said to be *maximal $\mathcal{M}_{sd}$-essential extension*, if any monomorphism g is an isomorphism, whenever gf is an $\mathcal{M}_{sd}$-essential extension.

(ii) By a *largest st-s-dense essential extension of A* we mean an $\mathcal{M}_{sd}$-essential extension $f : A \to B$ such that for each $\mathcal{M}_{sd}$-essential extension $g : A \to C$ there exists a monomorphism $h : C \to B$ such that $hg = f$.

Proposition 2.8. *Every right S-act has a maximal $\mathcal{M}_{sd}$-essential extension.*

Theorem 2.9. *An S-act B is a maximal $\mathcal{M}_{sd}$-essential extension of A if and only if it is a largest $\mathcal{M}_{sd}$-essential extension of A.*

3. Characterization of $\mathcal{M}_{sd}$-essential extensions

Notation 3.1. For an S-act A and $a \in A$ we denote the homomorphism $f : \mathbf{S} \to A$, given by $f(s) = as$ for all $s \in S$, by λ_a. Also the set $\{\lambda_a : a \in A\}$ is denoted by $\lambda(A)$. It is clear that $\lambda(A)$ is a subact of the act $Hom(\mathbf{S}, A)$, of all homomorphisms from $\mathbf{S}$ to A, with the actions given by $(fs)(t) = f(st)$, for $s, t \in S$. Also the assignment $\tau : a \mapsto \lambda_a$ is a homomorphism. We denote $\lambda_F(A)$ to be a subact of all homomorphism $f \in Hom(\mathbf{S}, A)$, where for every finite subset T of S, there exists $a_T \in A$ such that $f(t) = a_T t$. It is clear that $\lambda(A)$ is a subact of $\lambda_F(A)$.

Theorem 3.2. (**Essential Test Lemma**) *An st-s-dense extension $f : A \to B$ is $\mathcal{M}_{sd}$-essential if and only if for each $b \in B \setminus f(A)$, $b' \in B$ if $\lambda_b \mid_J = \lambda_{b'} \mid_J$ for $J = I_b \cup I_{b'}$, then $b = b'$.*

Definition 3.3. Let A be an S-act. We call an extension B of A a $\bar{\Gamma}$-*extension* of A if there is an isomorphism between B and the S-act $A(\bar{\Gamma}) = A \cup \bar{\Gamma}$, for some subset $\bar{\Gamma}$ of $\lambda_F(A) \setminus \lambda(A)$, which maps A identically.

For the case where A is separated, $A(\bar{\Gamma}) \cong \lambda_F(A)$ for $\bar{\Gamma} = \lambda_F(A) \setminus \lambda(A)$. This is because $A \cong \lambda(A)$, as stated in Notation 3.1.

Theorem 3.4. (**Characterization of $\mathcal{M}_{sd}$-essential extensions**) *An S-act B is an $\mathcal{M}_{sd}$-essential extension of A if and only if it is a $\bar{\Gamma}$-extension of A.*

4. The well-behaviour of st-s-dense injectivity

Recall the three so called Well-Behaviour Theorems of injectivity mentioned in Section 1. In this section we study these theorems for st-s-dense injectivity and show that they hold for acts over any semigroup S.

Definition 4.1. A subact A of an S-act B is said to be a *retract* of B if there exists a homomorphism (necessarily epimorphism) $\pi : B \to A$ such that $\pi(a) = a$ for all $a \in A$. An S-act A is said to be *st-s-dense absolute retract* if it is a retract of each of its st-s-dense extensions.

Now we prove the First Theorem of Well-Behaviour of Injectivity for st-s-dense injectivity. First recall that an S-act A is st-s-dense injective, if A is injective with respect to st-s-dense monomorphisms(i.e, every homomorphism $f : B \to A$ can be lifted to each st-s-dense extension C of B, which acts the same as f on B).

Theorem 4.2. (First Theorem of Well-Behaviour) *For a semigroup S and an S-act A, the following are equivalent:*

(i) *A is st-s-dense injective.*

(ii) *A is st-s-dense absolute retract.*
(iii) *A has no proper st-s-dense essential extension.*

Definition 4.3. An S-act B is said to be *strongly s-dense injective hull* or *st-s-dense injective hull* of an S-act A if it is an st-s-dense essential extension of A which is st-s-dense injective.

Theorem 4.4. *An S-act B is a maximal st-s-dense essential extension of A if and only if it is an st-s-dense injective hull of A.*

Theorem 4.5. (Second Theorem of Well-Behavior) *For a semigroup S, each S-act has a unique st-s-dense injective hull.*

Theorem 4.6. *Let A be an S-act, and $\bar{\Gamma} = \lambda_F(A) \setminus \lambda(A)$. Then the S-act $A(\bar{\Gamma})$ is an st-s-dense injective hull of A.*

Definition 4.7. An $\mathcal{M}_{sd}$-injective $\mathcal{M}_{sd}$-extension $f : A \to B$ of A is said to be *minimal* if any $\mathcal{M}_{sd}$-morphism $g : C \to B$ from an $\mathcal{M}_{sd}$-injective $\mathcal{M}_{sd}$-extension $h : A \to C$ to B (that is, with $gh = f$) is an isomorphism.

Theorem 4.8. *Let $f : A \to B$ be an $\mathcal{M}_{sd}$-injective $\mathcal{M}_{sd}$-extension of A. The homomorphism f is minimal if and only if for each $\mathcal{M}_{sd}$-injective $\mathcal{M}_{sd}$-extension $h : A \to C$ of A there exists a monomorphism $g : B \to C$ such that $gf = h$.*

Theorem 4.9. (Third Theorem of Well-Behavior) *For any extension B of an S-act A, the following are equivalent:*
(i) *B is an st-s-dense injective hull of A.*
(ii) *B is a maximal st-s-dense essential extension of A.*
(iii) *B is a minimal st-s-dense injective st-s-dense extension of A.*

Putting together the above results, we get:

Theorem 4.10. *For a semigroup S, $\mathcal{M}_{sd}$-injectivity well-behaves in the category* **Act-S**.

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CENTERLESS CORES OF LOCALLY EXTENDED AFFINE LIE ALGEBRAS

G. BEHBOODI ESKANDARI

ABSTRACT. In this work, we give a "direct union" realization of (centerless cores) of locally extended affine Lie algebras using the existing realizations of extended affine Lie algebras. We expect this would lead to a complete realization of (centerless cores of) locally extended affine Lie algebras.

This is a joint work with S. Azam and M. Yousofzadeh.

1. INTRODUCTION

There exists an effective framework for realization of extended affine Lie algebras (cf.[AABGP, §III]). In the present work, using this realization and by means of "direct unions", we provide a framework for realization of locally extended affine Lie algebras.

To realize extended affine Lie algebras, the authors in [AABGP] construct an extended affine Lie algebra whose centerless core is isomorphic to a Lie algebra $\mathscr{G}$ satisfying certain 11 axioms. One of the axioms defining $\mathscr{G}$ is that $\mathscr{G}$ has a weight space decomposition with respect to a subalgebra whose corresponding root system is an irreducible finite root system. In this paper we generalize the mentioned 11 axioms by switching from finite root systems to locally finite root systems and then study the class $\mathscr{T}$ consisting of Lie algebras satisfying these new 11 axioms. This allows us to realize (centerless cores of) locally extended affine Lie algebras introduced several years ago by Morita and Yoshii [MY] . We show that each element of $\mathscr{T}$ is a direct union of Lie subalgebras satisfying the former 11 axioms. We refer the interested readers to [A], [AG], [AKY], [K], [N1, N2, N3] for some other related works.

2. MAIN RESULTS

Throughout this work $\mathbb{F}$ is a field of characteristic zero.

Definition 2.1. Let $\mathscr{G}$ be a Lie algebra over $\mathbb{F}$ and $\mathscr{H}$ be a subalgebra of $\mathscr{G}$. We call $\mathscr{H}$ a *toral* subalgebra or an *ad-diagonalizable* subalgebra if

$$\mathscr{G} = \sum_{\alpha \in \mathscr{H}^*} \mathscr{G}_\alpha(\mathscr{H}) \tag{1}$$

where for any $\alpha \in \mathscr{H}^*$ (the dual space of $\mathscr{H}$),

$$\mathscr{G}_\alpha(\mathscr{H}) := \{x \in \mathscr{G} \mid [h,x] = \alpha(h)x \quad \text{for all} \quad h \in \mathscr{H}\}.$$

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In this case, $(\mathscr{G},\mathscr{H})$ is called a *toral pair* and the decomposition (1) is called the *root space decomposition* of $\mathscr{G}$ with respect to $\mathscr{H}$. Also $\alpha \in \mathscr{H}^*$ is called a *root* if $\mathscr{G}_\alpha(\mathscr{H}) \neq \{0\}$ and $R := \{\alpha \in \mathscr{H}^* \mid \mathscr{G}_\alpha(\mathscr{H}) \neq \{0\}\}$ is called the *root system* of $\mathscr{G}$ with respect to $\mathscr{H}$. We will usually abbreviate $\mathscr{G}_\alpha(\mathscr{H})$ by $\mathscr{G}_\alpha$. Since any toral subalgebra is abelian, $\mathscr{H} \subseteq \mathscr{G}_0$ and so $0 \in R$ unless $\mathscr{H} = \{0\} = \mathscr{G}$.

Definition 2.2. [LN, Definition 3.3] Let $\mathscr{V}$ be a nontrivial vector space over $\mathbb{F}$ and R be a subset of $\mathscr{V}$. R is said to be a *locally finite root system in $\mathscr{V}$ of rank dim($\mathscr{V}$)* if the following are satisfied:

(i) R is locally finite, contains zero and spans $\mathscr{V}$.

(ii) For every $\alpha \in R\backslash\{0\}$, there exists $\check{\alpha} \in \mathscr{V}^*$ such that $\check{\alpha}(\alpha) = 2$ and $s_\alpha(\beta) \in R$ for $\alpha,\beta \in R$ where $s_\alpha : \mathscr{V} \to \mathscr{V}$ maps $u \in \mathscr{V}$ to $u - \check{\alpha}(u)\alpha$. We set by convention $\check{0}$ to be zero.

(iii) $\check{\alpha}(\beta) \in \mathbb{Z}$, for $\alpha,\beta \in R$.

The locally finite root system R is also denoted by $(R,\mathscr{V})$. Suppose that R is a locally finite root system in $\mathscr{V}$. The subgroup of automorphisms of $\mathscr{V}$ generated by the set $\{s_\alpha \mid \alpha \in R\}$ is called the *Weyl group of R* and denoted by $\mathscr{W}_R$. We say two nonzero roots α,β are *connected* if there exist finitely many roots $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n = \beta$ such that $\check{\alpha}_{i+1}(\alpha_i) \neq 0, 1 \leq i \leq n-1$. A nonempty subset X of R is called *irreducible*, if each two nonzero elements $x,y \in X$ are connected. If the cardinal of a locally finite root system R is finite, it is called a *finite root system.*

Suppose that $\mathscr{T}$ is the class of all 4-tuples $(\mathscr{G},(\cdot,\cdot),\mathscr{H},G)$ satisfying the following axioms:

(1) $\mathscr{G}$ is a Lie algebra and $(\cdot,\cdot)$ is an invariant symmetric bilinear form on $\mathscr{G}$, where by invariant we mean $([x,y],z) = (x,[y,z])$ for all $x,y,z \in \mathscr{G}$,

(2) $\mathscr{H}$ is a nontrivial subalgebra of $\mathscr{G}$ such that $(\mathscr{G},\mathscr{H})$ is a toral pair with corresponding root system R,

(3) $(\cdot,\cdot)|_{\mathscr{H}\times\mathscr{H}}$ is nondegenerate; in particular the map $\mathfrak{b} : \mathscr{H} \longrightarrow \mathscr{H}^*$ mapping h to $(h,\cdot)$ is injective and so we can transfer the form $(\cdot,\cdot)$ on $\mathscr{H}$ to a form, denoted again by $(\cdot,\cdot)$, on $\mathfrak{b}(\mathscr{H})$ in a natural way,

(4) $\mathscr{G}$ is generated as a Lie algebra by $\sum_{\alpha\in R\backslash\{0\}} \mathscr{G}_\alpha$,

(5) G is an abelian group and $\mathscr{G} = \oplus\sum_{\sigma\in G}\mathscr{G}^\sigma$ is a G-graded Lie algebra. We refer to G as the grading group of $\mathscr{G}$,

(6) $\langle\{\sigma \in G \mid \mathscr{G}^\sigma \neq \{0\}\}\rangle = G$,

(7) $\sigma,\tau \in G,\ \sigma+\tau \neq 0 \implies (\mathscr{G}^\sigma,\mathscr{G}^\tau) = \{0\}$,

(8) for $\alpha \in R$, $\mathscr{G}_\alpha = \sum_{\sigma\in G}\mathscr{G}_\alpha^\sigma$ where $\mathscr{G}_\alpha^\sigma := \mathscr{G}_\alpha \cap \mathscr{G}^\sigma$ for $\alpha \in R$ and $\sigma \in G$,

(9) if $\mathscr{G}_\alpha^\sigma \neq \{0\}$ for some $\alpha \in R\setminus\{0\}$ and $\sigma \in G$, then $[\mathscr{G}_\alpha^\sigma, \mathscr{G}_{-\alpha}^{-\sigma}] \neq \{0\}$, and $\mathscr{G}_\alpha^0 \neq \{0\}$ for each $\alpha \in R$ such that $\frac{1}{2}\alpha \notin R$,

(10) $\mathscr{H} = \mathscr{G}_0^0$, in particular this together with (1), (3) and (9) implies that $R \subseteq \mathfrak{b}(\mathscr{H})$,

(11) R is a locally finite root system in its $\mathbb{F}$-span such that the restriction of the form $(\cdot,\cdot)$ to the subspace of $\mathscr{H}^*$ spanned by R is invariant under the Weyl group $\mathscr{W}_R$.

In [N3], the author considers extended affine Lie algebras and their generalizations, including locally extended affine Lie algebras and invariant affine reflection algebras. It is clarified there that the concept of the "core", the subalgebra generated by root spaces corresponding to nonisotropic roots, plays a central role in the theory. To describe the core, in [N3, §5] the concept of an *invariant predivison-(R,G)-graded* Lie algebra is defined, G an abelain group and R a locally finite root system, and it is shown that if G is torsion free then every invariant predivision-(R,G)-graded Lie algebra arises as the centerless core (i.e. core modulo center) of an invariant affine reflection algebra (see [N3, Theorem 6.10]). Now one sees from the definition of the class $\mathscr{T}$ that each element $(\mathscr{G},(\cdot,\cdot),\mathscr{H},G)\in\mathscr{T}$ is a predivision-(R,G)-graded Lie algebra and moreover, if $(\cdot,\cdot)$ is nondegenerate on $\mathscr{G}$, it can be seen from [N3, §5.1] that $\mathscr{G}$ is in fact an invariant predivison-(R,G)-graded Lie algebra. Therefore, when G is torsion free and the form $(\cdot,\cdot)$ is nondegenerate, each element of $\mathscr{T}$ can be regarded as centerless core of an invariant affine reflection algebra. Moreover, the relation between locally extended affine Lie algebras and invariant affine reflection algebras is discussed in [N3, §6.17]. In particular it is shown that locally extended affine Lie algebras are precisely invariant affine reflection algebras for which the corresponding toral subalgebra is self centralizing, and the set of nonisotropic roots is indecomposable. It turns out that when G is torsion free and the form is nondegenerate, each element of the class $\mathscr{T}$ corresponds to the centerless core of a locally extended affine Lie algebra.

Our main goal in the present work is to show that when R is a locally finite root system, then each 4-tuple in $\mathscr{T}$ can be characterized as a direct union of objects in $\mathscr{T}$ for which their corresponding root systems are finite root systems and their grading groups are finitely generated. In fact we have the following proposition.

Proposition 2.3. *Let $(\mathscr{G},(\cdot,\cdot),\mathscr{H},G)\in\mathscr{T}$ with corresponding root system R. Assume that R is a locally finite root system in its $\mathbb{F}$-span. Then there is a class $\{(\mathscr{G}_{(\lambda)},(\cdot,\cdot)_\lambda,\mathscr{H}_\lambda,G_\lambda)\}_{\lambda\in\Lambda}$ contained in $\mathscr{T}$ where Λ is an index set and for each $\lambda\in\Lambda$, $(\cdot,\cdot)_\lambda:=(\cdot,\cdot)|_{\mathscr{G}_{(\lambda)}\times\mathscr{G}_{(\lambda)}}$, G_λ is a finitely generated subgroup of G and $\mathscr{G}_{(\lambda)}$ is a Lie subalgebra of $\mathscr{G}$ and also $\mathscr{G}$, $\mathscr{H}$ and G are the direct unions of $\{\mathscr{G}_{(\lambda)}\}_{\lambda\in\Lambda}$, $\{\mathscr{H}_\lambda\}_{\lambda\in\Lambda}$ and $\{G_\lambda\}_{\lambda\in\Lambda}$ respectively. Moreover, for each $\lambda\in\Lambda$, the corresponding root system of $\mathscr{G}_{(\lambda)}$ with respect to $\mathscr{H}_\lambda$ is a finite root system in its $\mathbb{F}$-span.*

We also prove a converse result which will be our main tool in realizing (centerless cores of) locally extended affine Lie algebras, see Proposition 2.4. Roughly speaking, we show that if a Lie algebra $\mathscr{G}$, equipped with a suitable form, is the direct union of a family of objects in $\mathscr{T}$ whose forms are inherited from the one on $\mathscr{G}$, then $(\mathscr{G},(\cdot,\cdot),\mathscr{H},G)\in\mathscr{T}$, where $\mathscr{H}$ and G are also the direct unions of the corresponding terms.

Proposition 2.4. *Let $(\mathscr{G},\mathscr{H})$ be a toral pair with corresponding root system R and $(\cdot,\cdot)$ a bilinear form on $\mathscr{G}$. Assume that R is a locally finite root system in its $\mathbb{F}$-span and $\mathscr{G}_0=\sum_{\alpha\in R\setminus\{0\}}[\mathscr{G}_\alpha,\mathscr{G}_{-\alpha}]$. Also assume that G is an additive abelian group, $\mathscr{G}=\sum_{\sigma\in G}\mathscr{G}^\sigma$ is a G-graded Lie algebra and $\mathscr{G}_\alpha=\sum_{\sigma\in G}\mathscr{G}_\alpha^\sigma$ for all $\alpha\in R$ with $\mathscr{G}_\alpha^\sigma:=\mathscr{G}_\alpha\cap\mathscr{G}^\sigma$. Then there is a class $\{(\mathscr{G}_{(\lambda)},(\cdot,\cdot)_\lambda,\mathscr{H}_\lambda,G_\lambda)\}_{\lambda\in\Lambda}$, Λ is an index set, in which for each $\lambda\in\Lambda$, $(\cdot,\cdot)_\lambda:=(\cdot,\cdot)|_{\mathscr{G}_{(\lambda)}\times\mathscr{G}_{(\lambda)}}$, the corresponding root system of $\mathscr{G}_{(\lambda)}$ with respect to $\mathscr{H}_\lambda$ is a finite root system in its $\mathbb{F}$-span, G_λ is a finitely generated subgroup of G and $\mathscr{G}_{(\lambda)}$ is a Lie subalgebra of $\mathscr{G}$ and also $\mathscr{G}$, $\mathscr{H}$ and G are the direct unions of $\{\mathscr{G}_{(\lambda)}\}_{\lambda\in\Lambda}$, $\{\mathscr{H}_\lambda\}_{\lambda\in\Lambda}$ and $\{G_\lambda\}_{\lambda\in\Lambda}$ respectively and if $(\mathscr{G}_{(\lambda)},(\cdot,\cdot)_\lambda,\mathscr{H}_\lambda,G_\lambda)\in\mathscr{T}$ for all $\lambda\in\Lambda$, so is $(\mathscr{G},(\cdot,\cdot),\mathscr{H},G)$.*

The mentioned results on the class $\mathscr{T}$ help us to provide new examples of (centerless cores of) locally extended affine Lie algebras. We expect this procedure leads to a complete realization of (centerless cores) of all locally extended affine Lie algebras.

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ZERO PRODUCT DETERMINED RINGS

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Abstract. In this article we introduce the concept of zero product determined rings and present some results concerning this notion.

This is a joint work with H. Ghahramani[‡].

1. Introduction

The question of characterizing linear (additive) maps that preserve zero products and derivable maps at zero point on algebras (rings) has attracted the attention of many authors (for instance, see [4], [7] and the references therein). These problems can be sometimes effectively solved by considering bilinear maps that preserve certain zero product properties (for instance, see [1]–[3]. Motivated by these reasons Brešar et al. [5] introduced the concept of zero product determined algebras, which can be used to study the linear maps preserving zero product and derivable maps at zero point. Bilinear maps preserving zero products on Banach algebras were studied systematically in [2] and the authors proved in [5] that the matrix algebra $M_n(\mathcal{B})$ of $n \times n$ matrices over a unital algebra B is zero product determined. Also Brešar in [6] showed that if a unital algebra $\mathcal{A}$ is generated by its idempotents, then $\mathcal{A}$ is zero product determined and in [8] and [9], the author studied whether nest algebras and triangular algebras are zero product determined. Also in [10] zero product determined rings are studied. The goal of this paper is to introduce the concept of zero product determined rings and present some results concerning this notion.

2. Main results

In this section we give definition and some properties of zero product determined rings. We begin with a definition.

Definition 2.1. Let $\mathcal{A}$ be a ring and $\mathcal{S}$ be an additive group. A map $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{S}$ is called *bi-additive* if

$$\phi(a+b,c) = \phi(a,c) + \phi(b,c) \quad \textit{and} \quad \phi(a,b+c) = \phi(a,b) + \phi(a,c)$$

for all $a, b, c \in \mathcal{A}$. The bi-additive map ϕ is called *zero-product preserving* if $\phi(a,b) = 0$ for $a, b \in \mathcal{A}$ with $ab = 0$.

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If $T : \mathcal{A} \to \mathcal{S}$ is an additive map, then the bi-additive map $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{S}$ given by $\phi(a, b) = T(ab)$ is an example of zero-product preserving map. In general this is not the only possible example of zero-product preserving bi-additive maps (some examples will be obtained). In next proposition we assume that every zero-product preserving bi-additive map is determined by an additive map as above and give some equivalent conditions with it.

Proposition 2.2. *Let $\mathcal{A}$ be a ring. Then the following conditions are equivalent:*

(i) *For each zero-product preserving bi-additive map $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{S}$, where $\mathcal{S}$ is an arbitrary additive group, there exists an additive map $T : \mathcal{A} \to \mathcal{S}$ such that $\phi(a, b) = T(ab)$ for all $a, b \in \mathcal{A}$.*

(ii) *Each zero-product preserving bi-additive map $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{S}$, where $\mathcal{S}$ is an arbitrary additive group, satisfies*

$$\phi(ab, c) = \phi(a, bc) \quad (a, b, c \in \mathcal{A}),$$

(iii) *Each zero-product preserving bi-additive map $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{S}$, where $\mathcal{S}$ is an arbitrary additive group, satisfies*

$$\phi(a, b) = \phi(ab, 1) \quad (a, b \in \mathcal{A}).$$

(iv) *Each zero-product preserving bi-additive map $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{S}$, where $\mathcal{S}$ is an arbitrary additive group, satisfies*

$$\phi(a, b) = \phi(1, ab) \quad (a, b \in \mathcal{A}).$$

(v) *For each zero-product preserving bi-additive map $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{S}$, where $\mathcal{S}$ is an arbitrary additive group, if $a_k, b_k \in \mathcal{A}$, $k = 1, \cdots, m$, are such that $\sum_{k=1}^{m} a_k b_k = 0$, then $\sum_{k=1}^{m} \phi(a_k, b_k) = 0$.*

The above proposition motivates the next definition.

Definition 2.3. A ring $\mathcal{A}$ is a *zero product determined ring* if the equivalent conditions of Proposition 2.2 are satisfied.

From Proposition 2.2 we get the next corollary.

Corollary 2.4. *Let $\mathcal{A}$ be a zero product determined commutative ring. Then every zero-product preserving bi-additive map $\phi : \mathcal{A} \times \mathcal{A} \to \mathcal{S}$, where $\mathcal{S}$ is any additive group, is symmetric; that is $\phi(a, b) = \phi(b, a) \quad (a, b \in \mathcal{A})$.*

In the next results we shall obtain several examples of zero product determined rings which by the following proposition, we can find various examples of rings which are not zero product determined. Recall that a ring $\mathcal{A}$ is called *semi-commutative* if for any $a, b \in \mathcal{A}$, $ab = 0$ implies $a\mathcal{A}b = \{0\}$.

Proposition 2.5. *Suppose that $\mathcal{A}$ is a semi-commutative and zero product determined ring. Then $\mathcal{A}$ is a commutative ring.*

This proposition shows that if $\mathcal{A}$ is a semi-commutative ring which is not commutative, then $\mathcal{A}$ is not a zero product determined ring. So non-commutative domains are not zero product determined, since each domain is a semi-commutative ring. In the following example we give classes of non-commutative rings which are semi-commutative but not domain. Recall that a ring $\mathcal{A}$ with no nonzero nilpotent elements is called *reduced*.

Example 2.6. Let $\mathcal{R}$ be a reduced ring. The following rings are semi-commutative rings which are not commutative.

(1)
$$\mathcal{A}_1 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \,\middle|\, a, b, c, d \in \mathcal{R} \right\}.$$

(2) Suppose that $\mathcal{R}$ is non-commutative. Set
$$\mathcal{A}_2 = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \,\middle|\, a, b \in \mathcal{R} \right\}.$$

Now we study some hereditary properties of zero product determined rings.

Proposition 2.7. *Let $\mathcal{A}$ be a zero product determined ring, $\mathcal{B}$ be a ring, and $\theta : \mathcal{A} \to \mathcal{B}$ be an epimorphism. Then $\mathcal{B}$ is a zero product determined ring.*

Proposition 2.8. *Let $\mathcal{A}$, $\mathcal{A}_1$,...,$\mathcal{A}_n$ be rings such that $\mathcal{A} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ (ring direct sum). Then $\mathcal{A}$ is a zero product determined ring if and only if each $\mathcal{A}_k$ $(1 \leq k \leq n)$ is a zero product determined ring.*

Proposition 2.9. *Let $\mathcal{A}$ be a zero product determined ring and $\mathcal{I}$ be a two-sided ideal of $\mathcal{A}$. Then:*

(i) *$\mathcal{A}/\mathcal{I}$ is a zero product determined ring.*

(ii) *Each zero-product preserving bi-additive map $\phi : \mathcal{I} \times \mathcal{I} \to \mathcal{S}$, where $\mathcal{S}$ is an arbitrary additive group, satisfies*
$$\phi(xa, y) = \phi(x, ay) \quad (x, y \in \mathcal{I} \, and \, a \in \mathcal{A}),$$
In particular, if $\mathcal{I}$ has identity, then $\mathcal{I}$ is a zero product determined ring.

We continue by studying the connection between zero product determined rings and the question of determining mappings through their action on the zero-product elements $a, b \in \mathcal{A}$ (i.e., $ab = 0$).

Proposition 2.10. *Let $\mathcal{A}, \mathcal{B}$ be rings and $\alpha, \beta : \mathcal{A} \to \mathcal{B}$ be additive maps such that $\alpha(a)\beta(b) = 0$ for all $a, b \in \mathcal{A}$ with $ab = 0$. If $\mathcal{A}$ is a zero product determined ring, then*
$$\alpha(1)\beta(ab) = \alpha(ab)\beta(1) = \alpha(a)\beta(b)$$
for all $a, b \in \mathcal{A}$. Moreover, if $\alpha(1) = \beta(1) = 1$, then $\alpha = \beta$ and α is a homomorphism.

Corollary 2.11. *Let $\mathcal{A}$ be a zero product determined ring, $\mathcal{B}$ be a ring and $\alpha : \mathcal{A} \to \mathcal{B}$ be an additive map such that $\alpha(a)\alpha(b) = 0$ for all $a, b \in \mathcal{A}$ with $ab = 0$. Then*
$$\alpha(1)\alpha(ab) = \alpha(a)\beta(b) \quad and \quad \alpha(1)\alpha(a) = \alpha(a)\alpha(1)$$
for all $a, b \in \mathcal{A}$. In particular, if $\alpha(1) = 1$, then α is a homomorphism.

Let $\mathcal{A}$ be a ring and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. Recall that an additive map $\delta : \mathcal{A} \to \mathcal{M}$ is said to be a *derivation* (or *generalized derivation*) if $\delta(ab) = \delta(a)b + a\delta(b)$ (or $\delta(ab) = \delta(a)b + a\delta(b) - a\delta(1)b$) for all $a, b \in \mathcal{A}$.

Proposition 2.12. *Let $\mathcal{A}$ be a zero product determined ring and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. Suppose that $\tau, \delta : \mathcal{A} \to \mathcal{M}$ are additive maps such that $a\tau(b) + \delta(a)b = 0$ for all $a, b \in \mathcal{A}$ with $ab = 0$. Then τ and δ are generalized derivations. Indeed, if $\tau(1) = \delta(1) = 0$, then τ and δ are derivations.*

Corollary 2.13. *Let $\mathcal{A}$ be a zero product determined ring, $\mathcal{M}$ be an $\mathcal{A}$-bimodule, and $\delta : \mathcal{A} \to \mathcal{M}$ be an additive map such that $a\delta(b)+\delta(a)b = 0$ for all $a, b \in \mathcal{A}$ with $ab = 0$. Then δ is a generalized derivation such that $a\delta(1) = \delta(1)a$ for each $a \in \mathcal{A}$. Indeed, if $\delta(1) = 0$, then δ is a derivation.*

Next, we continue to give some classes of the zero product determined rings. Recall that the ring $\mathcal{A}$ is *simple* in the case where $\{0\}$ and $\mathcal{A}$ are the only ideals of $\mathcal{A}$.

Theorem 2.14. *Any simple ring containing a nontrivial idempotent is a zero product determined ring.*

Let $M_n(\mathcal{R})$ be the ring of all $n \times n$ matrices over a ring $\mathcal{R}$.

Theorem 2.15. *If $\mathcal{R}$ is a unital ring, then $M_n(\mathcal{R})$ is a zero product determined ring for every $n \geq 2$.*

Recall that a *triangular ring* $Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is a ring of the form

$$Tri(\mathcal{A}, \mathcal{M}, \mathcal{B}) := \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \middle| a \in \mathcal{A}, b \in \mathcal{B}, m \in \mathcal{M} \right\}$$

under the usual matrix operations, where $\mathcal{A}$ and $\mathcal{B}$ are rings and $\mathcal{M}$ is an $(\mathcal{A}, \mathcal{B})$-bimodule. Let $\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ and $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then e is a non-trivial idempotent in $\mathcal{T}$ and $f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ such that $e\mathcal{T}f\mathcal{T}e = \{0\}$ and $f\mathcal{T}e\mathcal{T}f = \{0\}$. Also $e\mathcal{T}e \cong \mathcal{A}$ and $f\mathcal{T}f \cong \mathcal{B}$ (ring isomorphisms). We have the following theorem which is a slight generalization of [9, Theorem 2.1.].

Theorem 2.16. *Let $\mathcal{T} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular ring. Then $\mathcal{T}$ is a zero product determined ring if and only if $\mathcal{A}$ and $\mathcal{B}$ are zero product determined rings.*

The following theorem is a generalization of [8, Proposition 2.4.].

Theorem 2.17. *Let $\mathcal{H}$ be a Hilbert space over $\mathbb{F}$, with $dim\mathcal{H} \geq 2$. Then $\mathcal{B}(\mathcal{H})$ is a zero product determined ring.*

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INFLUENCE CAP*-SUBALGEBRAS ON STRUCTURE OF LIE ALGEBRAS

SARA CHEHRAZI

Abstract. A subalgebra H of a Lie algebra L is said a CAP*-subalgebra if, for any non-Frattini chief factor A/B of L, we have $H + A = H + B$ or $H \cap A = H \cap B$. In this talk, some new results are obtained based on the assumption that some subalgebras is the CAP*-subalgebra in the Lie algebra. Also, the connection between the structure of a Lie algebra and its CAP*-subalgebras are discussed and some sufficient conditions are obtained for a Lie algebra being solvable or supersolvable.

1. Introduction

All Lie algebras considered in this paper are finite dimensional over an arbitrary field F. Let H be a subalgebra of L, the following are notations which will be used:

H_L is the core (with respect to L) of H, that is the largest ideal of L contained in H; $N(L)$ denotes the nil radical of L; $\varphi(L)$ is the Frattini ideal of L, that is the core of Frattini subalgebra of L (intersection of all maximal subalgebras of L); H is called a 2-maximal subalgebra of L if H is a maximal subalgebra of some maximal subalgebra M of L; If B is an ideal of L, we will write $B \trianglelefteq L$. In addition, $M <\cdot L$ indicates that M is a maximal subalgebra of L; If A and B are subalgebras of L for which $L = A+B$ and $A \cap B = 0$, we will write $L = A\dot{+}B$.

A subalgebra H of a Lie algebra L is said to have the cover-avoiding property in L if H either covers or avoidance every chief factor of L, in short, H is a CAP-subalgebra of L. The CAP-subalgebras are studied by many authers, for example, Hallahan and Overbeck ([1]), and Stitzinger ([2]) that found some kind of CAP-subalgebras in a solvable or metanilpotent Lie algebra. In [3], Towers used this concept to investigate the structure of a Lie algebra.

Theorem 1.1. (cf.[3, Corollary 2.8]) Every one-dimensional subalgebra of L is a CAP-subalgebra of L if and only if L is supersolvable.

In this talk, we define a new subalgebra of L as the following:

Definition 1.2. A subalgebra H of a Lie algebra L is said a CAP*-subalgebra if, for any non-Frattini chief factor A/B of L, we have $H + A = H + B$ or $H \cap A = H \cap B$.

We discuss the connection between the structure of a Lie algebra and its CAP*-subalgebras and we obtain some sufficient conditions for a Lie algebra being solvable or supersolvable.

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Remark 1.3. Clearly a CAP-subalgebra of L is a CAP*-subalgebra of L. Howerver, the converse is not true, as the following example shows.

Example 1.4. Let $L = \mathbb{R}a_1 + \mathbb{R}a_2 + \mathbb{R}a_3 + \mathbb{R}a_4$ be a Lie Algebra with the multiplication defined by $[a_3, a_1] = a_2$, $[a_2, a_3] = a_1$ and $[a_4, a_3] = a_1$, other products being zero. An easy proof gives $\varphi(L) = \mathbb{R}a_1 + \mathbb{R}a_2$ is a minimal ideal of L. It follows that all chief factors of L are non-Frattini besides $\varphi(L)/0$. Let $H = \mathbb{R}a_2$. It is easy to see that H avoids every non-Frattini chief factors of L. That is H is a CAP*-subalgebra of L. However, $H + \varphi(L) = \varphi(L) \neq H + 0$ and $H \cap \varphi(L) = H \neq 0 = H \cap 0$. Thus H is not a CAP-subalgebra of L since $\varphi(L)/0$ is a chief factor of L.

2. The Main Result

In this section, we give some characterizations of solvable and supersolvable Lie algebras.

Theorem 2.1. *Let L be a Lie algebra over a field F with at least* $\dim L$ *elements, such that all maximal subalgebras of L which contains idealiser of a maximal nilpotent subalgebra of L, are CAP*-subalgebras of L. Then L is solvable.*

Corollary 2.2. *Let L be a Lie algebra over a field F with at least* $\dim L$ *elements, such that all maximal subalgebras of L which contains a cartan subalgebra of L, are CAP*-subalgebras of L. Then L is solvable.*

Theorem 2.3. *Let M be a maximal subalgebra of a Lie algebra L. Then M is a CAP*-subalgebra of L if and only if $\eta(L : M) = dim(L/M)$.*

Corollary 2.4. (cf.[4, Corollary 2.6]) The Lie algebra L is solvable if and only if $\eta(L : M) = dim(L/M)$ for all maximal subalgebras M of L.

Theorem 2.5. *Let L be an arbitrary Lie algebra, $H \trianglelefteq L$ and*

$$F_h(L) = \{M | M < \cdot L, L = M + H\}.$$

If each maximal subalgebra which is contained in $F_h(L)$, be a CAP*-subalgebra of L, then H is solvable.

Corollary 2.6. *If all maximal subalgebras of a Lie algebra L are CAP*-subalgebras of L, then L is solvable.*

Note that the converse of Corollary 2.6 is true, by [3, Theorem 3.3].

Theorem 2.7. *Let L be a Lie algebra over a field F which has characteristic zero. Then L is solvable if and only if there exists a maximal subalgebra M of L such that M is a solvable CAP*-subalgebra of L.*

Theorem 2.8. *If every maximal nilpotent subalgebra of a Lie algebra L over a field F of characteristic zero, is a CAP*-subalgebra of L, then L is solvable.*

Corollary 2.9. *If every cartan subalgebra of a Lie algebra L over a field F of characteristic zero, is a CAP*-subalgebra of L, then L is solvable.*

Theorem 2.10. *If every 2-maximal subalgebra of a Lie algebra L is a CAP^*-subalgebra of L, then exactly one of the following conditions holds:*

1) L is solvable, or

2) L is simple, and every maximal subalgebra of L is one-dimensional; in particular, if F is perfect and of characteristic zero or $p > 3$, then L is three-dimensional simple and $\sqrt{F} \nsubseteq F$.

Theorem 2.11. *Let L be a Lie algebra over a field F of characteristic zero. If there exists a solvable 2-maximal subalgebra of L such that is a CAP^*-subalgebra of L, then exactly one of the following conditions holds:*

1) L is solvable,or

2) L is three-dimensional simple and $\sqrt{F} \nsubseteq F$.

Theorem 2.12. *Let L be a solvable Lie algebra over a field F of characteristic zero. If every maximal subalgebra of $N(L)$ is a CAP^*-subalgebra of L, then L is supersolvable.*

Corollary 2.13. *Let L be a solvable Lie algebra over a field of characteristic zero, in which every maximal subalgebra of $N(L)$ is a CAP-subalgebra of L, then L is supersolvable.*

Theorem 2.14. *Let H be a solvable ideal of a Lie algebra L such that L/H is supersolvable. If every maximal subalgebra of $N(H)$ is a CAP^*-subalgebra of L, then L is supersolvable.*

Corollary 2.15. *Let H be a solvable ideal of a Lie algebra L such that L/H is supersolvable. If every maximal subalgebra of $N(H)$ is a CAP-subalgebra of L, then L is supersolvable.*

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THE CONCEPTS OF REGULAR INJECTIVITY AND REGULAR PROJECTIVITY AND THEIR RELATIONS TO WEAK FACTORIZATION SYSTEMS

FARIDEH FARSAD[†]

Abstract. Let S be a pomonoid. In this talk, the category of S-posets and S-poset maps, **Pos**-S, is considered. First we recall the concept of weak factorization system in any given category and then we present some weak factorization systems in **Pos**-S. Also we investigate the relationship between this notion and known concepts such as regular injectivity and projectivity.

This is a joint work with Ali Madanshekaf[‡].

1. Introduction

Recall that a *pomonoid* is a monoid with a partial order $\leq$ which is compatible with the monoid operation: for $s, t, s', t' \in S$, $s \leq t$, $s' \leq t'$ imply $ss' \leq tt'$.

Let S be a pomonoid. A (*right*) *S-poset* is a poset A which is also an S-act whose action $\mu : A \times S \to A$ is order-preserving, where $A \times S$ is considered as a poset with componentwise order. The category of all S-posets with action preserving monotone maps between them is denoted by **Pos**-S. Clearly S itself is an S-poset with its operation as the action. Also, Let B be a non-empty subposet of A. Then B is called a sub S-poset of A if $bs \in B$ for all $s \in S$ and $b \in B$. For more information see [3].
Let $\mathbf{C}$ be a category and $\mathcal{H}$ a class of its morphisms. An object I of $\mathbf{C}$ is called $\mathcal{H}$-injective if for each $\mathcal{H}$-morphism $h : U \to V$ and morphism $u : U \to I$ there exists a morphism $s : V \to I$ such that $sh = u$. That is, the following diagram is commutative:

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "h"'] & I \\ V \arrow[ur, dashed, "s"'] & \end{tikzcd}$$

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In particular, this means that, in the slice category $\mathbf{C}/B$, $f : X \to B$ is $\mathcal{H}$-injective if, for any commutative diagram

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ V \arrow[r, "v"'] & B \end{tikzcd}$$

with $h \in \mathcal{H}$, there exists an arrow $s : V \to X$ such that $sh = u$ and $fs = v$.

The concept of weak factorization systems plays a central role in homotopy theory, in particular in the basic definition of Quillen model categories. Formally, this notion generalizes factorization systems by weakening the unique diagonalization property to the diagonalization property without uniqueness. Now, we introduce from [1] the notion which we deal with in this paper. We denote by $\Box$ the relation *diagonalization property* on the class of all morphisms of a category $\mathbf{C}$: given morphisms $l : A \to B$ and $r : C \to D$ then

$$l \Box r$$

means that in every commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "l"'] & C \arrow[d, "r"] \\ B \arrow[r] \arrow[ur, dashed, "d"] & D \end{tikzcd}$$

there exists a diagonal $d : B \to C$ rendering both triangles commutative. In this case l is also said to have the *left lifting property* with respect to r (and r to have the *right lifting property* with respect to l).

Let $\mathcal{H}$ be a class of morphisms. We denote by

$$\mathcal{H}^{\Box} = \{r \mid r \text{ has the right lifting property with respect to each } l \in \mathcal{H}\}$$

and

$$^{\Box}\mathcal{H} = \{l \mid l \text{ has the left lifting property with respect to each } r \in \mathcal{H}\}.$$

Let $\mathcal{H}_B$ be the class of those morphisms in $\mathbf{C}/B$ whose underlying morphism in $\mathbf{C}$ lies in $\mathcal{H}$. Now, $r : A \to B \in \mathcal{H}^{\Box}$ if and only if r is an $\mathcal{H}_B$-injective object in $\mathbf{C}/B$. Dually, all morphisms in $^{\Box}\mathcal{H}$ are characterized by a projectivity condition in $\mathcal{H}_B$.

2. Main Results

Recall from [1] that a *weak factorization system* in a category is a pair $(\mathcal{L}, \mathcal{R})$ of morphism classes such that
(1) every morphism has a factorization as an $\mathcal{L}$-morphism followed by an $\mathcal{R}$-morphism.
(2) $\mathcal{R} = \mathcal{L}^{\Box}$ and $\mathcal{L} = {}^{\Box}\mathcal{H}$.

Remark 2.1. If we replace "$\Box$" by "$\perp$" where "$\perp$" is defined via the *unique diagonalization property* (i.e., by insisting that there exists precisely one diagonal), we arrive at the familiar notion of a factorization system in a category. Factorization systems are weak factorization systems. For instance, let $\mathcal{E}_{\mathcal{S}}$ be the class of all S-poset epimorphisms. Then, by Theorem 1 of [3] one can easily seen that $(\mathcal{E}_{\mathcal{S}}, Emb)$ in $\mathbf{Pos}$-S is a factorization system.

Example 2.2. In the category **Pos** (of all partially ordered sets (posets) with order preserving (monotone) maps) pair (Emb, Top) is a weak factorization system, where Emb is the class of all order-embeddings; that is, S-poset maps $f : A \to B$ for which $f(a) \leq f(a')$ if and only if $a \leq a'$, for all $a, a' \in A$ and Top is the class of all topological isotone maps.

Recall that regular monomorphisms (morphisms which are equalizers) in **Pos**-S (and also in **Pos**-S/B_S) are exactly order-embeddings (see [3] and ([4])). In the following we consider Emb-injectivity in **Pos**-S and **Pos**-S/B_S, where Emb is the class of all order-embeddings of S-posets. In the following, we try to provide a weak factorization system for **Pos**-S with Emb as the left part.

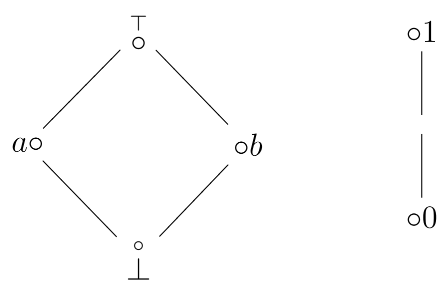
Definition 2.3 ([7])**.** A possibly empty sub S-poset A of an S-poset B is said to be *down-closed* in B if for each $a \in A$ and $b \in B$ with $b \leq a$ we have $b \in A$. By a *down-closed* embedding, we mean an embedding $f : A \to B$ such that $f(A)$ is a down closed sub S-poset of B.

Lemma 2.4. *Let S be a pomonoid whose identity e is the bottom element and $f : X \to Y$ be an S-poset map. If f is a down-closed embedding then* $\mathrm{im}(f)$ *is a direct summand of Y.*

Proposition 2.5. *Let S be a pomonoid. Suppose $f : A \to B$ is an Emb-injective object in* **Pos**-S/B *for any S-poset B. Then f is a split epimorphism.*

Corollary 2.6. *Suppose $f : A \to B$ is an Emb-injective object in* **Pos**-S/B. *If A is a complete lattice which is a retract of $A^{(S)}$, then A and B are Emb-injective in* **Pos**-S.

Remark 2.7. There exists a split epimorphism in **Pos**-S which is not Emb-injective in **Pos**-S/B. To present an example, take an arbitrary pomonoid S and X, B are two lattices as shown in the following



Then X is an S-poset with the action defined by $\top s = \top$ and $as = bs = \perp s = a$ for all $s \in S$, also we consider B with the trivial action as an S-poset. Define the S-poset map $f : X \to B \in \mathbf{Pos} - S/B$, by $f(a) = f(b) = f(\perp) = 0$ and $f(\top) = 1$. It also is a convex map. We show that it is not a regular injective object in **Pos**-S/B. Since $f^{-1}(0) = \{\perp, a, b\}$ is not a complete lattice, the authors in [4], showed that it is not Emb-injective in **Pos**-S/B.

On the other hands, define the S-poset map $g : B \to X \in$**Pos**-S/B, by $g(1) = \top, g(0) = g(\perp)$. Then we have $fg = \mathrm{id}_B$, so f is split epimorphism. Therefore, the converse of the above theorem is not true generally.

Recall that epimorphisms in **Pos**-S are exactly onto S-poset maps (see [3]). Applying this fact and the well-known adjunction $\sum_B \dashv \prod_B$ where the functors

$$\sum\nolimits_B : \mathbf{Pos} - S/B \to \mathbf{Pos} - S \quad \text{and} \quad \prod_B : \mathbf{Pos} - S \to \mathbf{Pos} - S/B$$

defined by the forgetful functor $\sum_B$: **Pos**-$S/B \to$ **Pos**-S which assigns to any f its domain and the pullback functor $\prod_B : \mathbf{Pos}-S \to \mathbf{Pos}-S/B$ which assigns the second projection $\pi_B^X : X \times B \to B$ to any S-poset X, we get the following sequel.

Theorem 2.8. *Suppose $f : A \to B$ is a projective object in* **Pos**-S/B. *Then A is projective in* **Pos**-S.

Theorem 2.9. *Let S be a pomonoid and $\mathcal{E}_\mathcal{S}$ is the class of all S-poset epimorphisms. Then $(\mathcal{U}_\mathcal{P}, \mathcal{E}_\mathcal{S})$ is a weak factorization system for* **Pos**-S *if and only if all S-posets have discrete order.*

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AN ALGORITHM FOR CALCULATING SPHERES IN SOME FAMILIES OF CAYLEY GRAPHS

MOHAMMAD HOSSEIN GHAFFARI†

ABSTRACT. Suppose that Ω is a generating set for a finite group G and g is an arbitrary element of G. Finding the minimum number of elements of Ω for producing g, is equivalent to the problem of finding the distance of g from the identity in Cayley graph $\Gamma = Cay(G, \Omega)$. To solve this problem it is enough to find the radius of the sphere in Γ on which g belongs. In this paper, we propose a useful algorithm to solve this problem for those Cayley graphs which are generated by all permutations of specified cycle structures.

This is a joint work with Zohreh Mostaghim‡.

1. Introduction

Let G be a finite group and let Ω be a set of generators of G such that the identity element e of G does not belong to Ω and $\Omega = \Omega^{-1}$, where $\Omega^{-1} = \{\omega^{-1} : \omega \in \Omega\}$. In the Cayley graph $\Gamma = Cay(G, \Omega) = (V, E)$ vertices correspond to the elements of the group, i.e. $V = G$, and edges correspond to the right multiplication by generators, i.e., $E = \{\{g, g\omega\} : g \in G, \omega \in \Omega\}$. Since $e \notin \Omega$, there is no loops in Γ. Also, Ω is required to be a generating set of G so that Γ is connected.

One of the problems in the group theory is finding the minimum number of generators needed to generate an element of a group (for instance, look at [2] and [7]). This problem is equivalent to finding the distance of an element from identity in a Cayley graph, that leads to the concept of *Word-Length*. With this notion, a group can be viewed as a metric space. In the same way as in the case of metric spaces we can define spheres and balls in a group.

Definition 1.1. Let Γ be a graph with vertex set V. We define the sphere and ball of radius r centered at c respectively as the following,

$$S(c, r) := \{ \ v \in V : d(c, v) = r \ \} \ , \qquad B(c, r) := \{ \ v \in V : d(c, v) \leq r \ \}.$$

Since by knowing the size of all spheres (as a set of vertices) in a graph, we can calculate the diameter of it; The problem of finding the spheres in graphs can be viewed as a generalization of the problem of finding the diameter of graphs. The diameter of Cayley graphs has been studied in many papers; For example, in [6] and

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[1] some bounds are founded for the diameter of some families of Cayley graphs. In [8] the diameter of Cayley graphs of subgroups of S_n generated by $k-$transpostions are calculated. In [3] the authors show that finding the diameter of a Cayley graph is NP-Hard[1].

2. Main Results

Suppose that G is a subgroup of S_n with generating set Ω, consists of all permutations of specified cycle structures; For example, all permutations of cycle structures $1^4 3^1$ and $1^2 2^1 3^1$. It is obvious that Ω is a union of some conjugacy classes and G is a normal subgroup of S_n.

Lemma 2.1. *For every sphere S in $\Gamma = Cay(G, \Omega)$, there is a set C such that the elements of S are all permutations with cycle structures in C.*

Since $S(c, r) = cS(e, r)$, by the above lemma, for calculating spheres of Γ, it is enough to calculate the cycle structure of elements of spheres centered at identity.

Based on the above results, we design an algorithm implemented in GAP,[2] to compute all spheres centered at identity by the command *GetSpheres*. The source code and sample outputs of *GetSpheres* is available online [5].

Theorem 2.2. *By the above notations, the implemented algorithm in GetSpheres command computes the cycle structure of permutations in the spheres of Γ by receiving the cycle structures of permutations of generating set Ω.*

In order to compare the performances of *GetSpheres* and GAP's algorithm, we present a table of runtime of them. The following table shows the required time (in milliseconds) for calculating the number of elements of spheres in the Cayley graph of the group generated by m-cycles in S_n. In fact *GetSpheres* (the fourth column) gives more; it finds cycle structures of all elements of spheres; the number of elements of spheres is just a conclusion of its calculations. The third column shows the runtime of

$$GrowthFunctionOfGroup(G)$$

where, G is defined for GAP as a group generated by m-cycle. Note that *GrowthFunctionOfGroup* can be used not only in our specialized case but for all Cayley graphs. The fifth column shows the diameter of the graph as a result of our computations.

[1]This notion in computational complexity theory points to the hardness of a problem.

[2]A system for computational discrete algebra, with particular emphasis on Computational Group Theory [4]

n	m	GAP	GetSpheres	Diameter
7	2	586	24	6
7	3	719	19	3
7	4	11890	236	4
8	2	6884	23	7
8	3	21234	67	4
8	4	188525	623	4
9	2	92416	36	8
9	3	175538	96	4
9	4	4082722	1408	5
10	2	-	105	9
10	3	-	213	5

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FUZZY NEXUS OVER AN ORDINAL

TOKTAM HAGHDADI†

Abstract. In this talk, we define fuzzy subnexuses over a nexus N. After that we define the notions of prime fuzzy subnexus and fraction induced by fuzzy subnexuses and we obtain some results about the above notions. Finally, we show that if S is a meet closed subset of $Fsub(N)$ and $h = \bigwedge S \in S$, then $S^{-1}N \cong \{h\}^{-1}N$ as meet-semilattice.

This is a joint work with Ali Akbar Estaji‡ and Javad Farrokhi Ostad††.

1. Introduction and Preliminaries

Fuzzy sets were introduced by Lotfi A. Zadeh [8] and Dieter Klaua [5] in 1965 as an extension of the classical notion of set. At the same time, Salii [7] defined a more general kind of structures called L-relations, which were studied by him in an abstract algebraic context. Fuzzy relations, which are used now in different areas, such as algebra and rough set.

In this paper the notion of a fuzzy subnexus and a prime fuzzy subnexus over an ordinal are defined. Then, we introduce notion fraction induced by nexuses and fuzzy subnexuses and we give some characterizations for fraction of N.

Let $\mathfrak{O}$ be the collection of all ordinal numbers and let $\gamma, \delta \in \mathfrak{O}$, $\gamma \geq 1$ and $\delta \geq 1$. An address over γ, is a function $a : \delta \to \gamma$, such that $a(i) = 0$ implies that $a(j) = 0$ for all $j \geq i$. We denote by $A(\gamma)$, the set of all address over γ.

Let $a : \delta \to \gamma$ be a address over γ. If for every $i \in \delta$, $a(i) = 0$, then it is called the empty address and denoted by (). If a is a nonempty address, then there exists a unique element $\beta \in \delta + 1$ such that for every $i \in \beta$, $a(i) \neq 0$ and for every $\beta \leq i \in \delta$, $a(i) = 0$, we denote this address by $(a_i)_{i\in\beta}$, where $a_i = a(i)$ for every $i \in \beta$.

The level of $a \in A(\gamma)$ is denoted by $l(a)$ and said to be:
1. $l(a) = 0$, if a=(). 2. $l(a) = \beta$, if $() \neq a = (a_i)_{i\in\beta}$.

Let a, b be two elements of $A(\gamma)$. Then we said to $a \leq b$ if $l(a) = 0$ or one of the following case satisfies for $a = (a_i)_{i\in\beta}$ and $b = (b_i)_{i\in\delta}$:

(1) If $\beta = 1$, then $a_0 \leq b_0$.
(2) If $\beta \geq 2$ is a non-limit ordinal, then $a|_{\beta-1} = b|_{\beta-1}$ and $a_{\beta-1} \leq b_{\beta-1}$.
(3) If β is a limit ordinal, then $a = b|_\beta$.

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A nexus N over γ is a set of addresses with the following properties:

(1) $\emptyset \neq N \subseteq A(\gamma)$.

(2) If $() \neq a = (a_i)_{i\in\beta} \in N$, then for every $\delta \in \beta$ and $0 \leq j \leq a_\delta$, $a^{(\delta,j)} \in N$.

For undefined terms and notation see [4].

2. Prime fuzzy nexuses and Fraction induced by fuzzy subnexuses

Definition 2.1. Let f be a fuzzy subset on nexus N. Then f is called a fuzzy subnexus of N, if $a \leq b$ implies that $f(b) \leq f(a)$ for all $a, b \in N$. The set of all fuzzy subnexus of N is denoted by $Fsub(N)$.

Proposition 2.2. *Let N be a nexuses over γ and $\{f_i\}_{i\in I} \subseteq Fsub(N)$, then*

$$1.\bigcup_{i\in I} f_i \in Fsub(N) \;\; 2.\bigcap_{i\in I} f_i \in Fsub(N)$$

Definition 2.3. Let N be nontrivial nexus over γ. Fuzzy subnexus f of N is called a prime fuzzy subnexus if, $f(a \wedge b) \leq max\{f(a), f(b)\}$, for all $a, b \in N$

Proposition 2.4. *Let N be a nontrivial nexsus over γ and f be a fuzzy subnexus of N. The following assertions are equivalent:*

(1) *f is prime fuzzy subnexus.*

(2) *For every $r \in [0, 1]$, if f_r is a nonempty subset N, then f_r is prime subnexus.*

(3) *For every $r \in [0, 1]$, $N \setminus f_r$ is closed under finite meet operation.*

Remark 2.5. Let $x \in N$ and $t \in (0, 1]$. Then $< x^t >: N \to [0, 1]$ defined by

$$< x^t > (a) = \begin{cases} t & x \in\uparrow a \\ \\ 0 & x \notin\uparrow a \end{cases}$$

is fuzzy subnexus.

Proposition 2.6. *Let N be a nexus and $f \in Fsub(N)$.*

(1) *If $|Imf| \leq 2$ and $\emptyset \neq f_* \in Psub(N)$, then $g \wedge h \subseteq f$ implies that $g \subseteq f$ or $h \subseteq f$, every $g, h \in Fsub(N)$.*

(2) *If $g \wedge h \subseteq f$ implies that $g \subseteq f$ or $h \subseteq f$, for every $g, h \in Fsub(N)$, then $|Imf| \leq 2$.*

(3) *If $|Imf| = 2$ and for every $g, h \in Fsub(N)$, $g \wedge h \subseteq f$ implies that $g \subseteq f$ or $h \subseteq f$, then $\emptyset \neq f_* \in Psub(N)$.*

Definition 2.7. A meet closed subset of $Fsub(N)$ is a nonempty subset S of $Fsub(N)$ such that $f \wedge g \in S$, for all $f, g \in S$.

Let S be a meet closed subset of $Fsub(N)$. Define the relation $\sim_S$ on $N \times S$ as follow:

$$(a, f) \sim_S (b, g) \Leftrightarrow \exists h \in S\, \forall t \in (0, 1](< a^t > \wedge g \wedge h =< b^t > \wedge f \wedge h).$$

We will proved that $\sim_S$ is an equivalence relation. The set of all equivalence classes $\sim_S$ on $N \times S$ is denoted by $S^{-1}N$.

Definition 2.8. Let S be a meet closed subset of $Fsub(N)$ and $\frac{a}{f}, \frac{b}{g} \in S^{-1}N$. Then we said to $\frac{a}{f} \leq \frac{b}{g}$, if there exists $h \in S$ such that, for every $t \in (0,1]$

$$< a^t > \wedge < b^t > \wedge f \wedge h = f \wedge g \wedge < a^t > \wedge h,$$

It is easy to show that $(S^{-1}N, \leq)$ is a meet-semilattice.

Proposition 2.9. *Let N be a nexus over γ and S be a meet closed subset of $Fsub(N)$.*

(1) *Every ideal of $S^{-1}N$ is the form of $S^{-1}I$, where I is a subnexus of N.*
(2) *If K is a finite ideal of $S^{-1}N$ and $h = \bigwedge S \in S$, then there exists a cyclic subnexus I of N such that $K = S^{-1}I$.*
(3) *If M be a prime ideal of $S^{-1}N$, then there exists $I \in Psub(N)$ such that $M = S^{-1}I$.*

Proposition 2.10. *Let S be a meet closed subset of $Fsub(N)$ and $h = \bigwedge S$. We define $\varphi : S^{-1}N \longrightarrow \{h\}^{-1}N$ with $\varphi(\frac{a}{f}) = \frac{a}{h}$. Then we have the following conclusions:*

(1) *φ is a onto meet-semilattice homomorphism.*
(2) *If $h \in S$, then φ is one to one. In particular, this shows if $h \in S$, then $S^{-1}N \cong \{h\}^{-1}N$ as meet-semilattice.*

Proposition 2.11. *Let N be a nexus over γ and let S be a meet closed subset of $Fsub(N)$. If $h = \bigwedge S$, then $\{h\}^{-1}N \cong \overbrace{\downarrow h}$ as meet-semilattice, where*

$$\overbrace{\downarrow h} = \{h \wedge < a^1 >; a \in N\}.$$

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QUASI HYPER BCK-ALGEBRAS

M. HAMIDI†

ABSTRACT. In this talk, by considering the concept of fundamental relation on quasi hyper BCK-algebras, we define the notions of quasi BCK-algebra and fundamental quasi BCK-algebras and prove that any countable quasi BCK-algebra is isomorphic to a fundamental quasi BCK-algebra. We show that no finite BCK-algebra is a fundamental quasi BCK-algebra over itself.

This is a joint work with R. A. Borzooei‡ and M. Mollaei††.

1. Introduction

The study of BCK-algebras was initiated by Y. Imai and K. Iseki [2] in 1966 as a generalization of the concept of set- theoretic difference and propositional calculi. In [3] Borzooei, et al. applied the hyperstructures to BCK-algebras, and introduced the concept of a hyper BCK-algebras which is a generalization of a BCK-algebra and investigated some related properties. Now, in this paper, we apply the fundamental relation β^* on quasi hyper BCK-algebras and generate an algebra as quasi BCK-algebra. Moreover, we introduce the notion of fundamental quasi BCK-algeras via the β^* and show that any countable quasi BCK-algebras are fundamental quasi BCK-algebras. But, we show that any finite quasi BCK-algebra is not a fundamental quasi BCK-algebra of itself.

2. **Preliminaries**

Definition 2.1. [2] Let X be a set with a binary operation " $*$ " and a constant "0". Then, $(X, *, 0)$ is called a *BCK-algebra* if it satisfies the following conditions:
(BCI-1) $((x * y) * (x * z)) * (z * y) = 0$,
(BCI-2) $(x * (x * y)) * y = 0$,
(BCI-3) $x * x = 0$,
(BCI-4) $x * y = 0$ and $y * x = 0$ imply $x = y$,
(BCK-5) $0 * x = 0$.
We define a binary relation " $\leq$ " on X by $x \leq y$ if and only if $x * y = 0$.

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Definition 2.2. [1] Let H be a nonempty set and $P^*(H)$ be the family of all nonempty subsets of H. A map $\circ : H \times H \longrightarrow P^*(H)$,is called binary hyperoperation.

Definition 2.3. [3] Let H be a non-empty set, endowed with a binary hyperoperation "$\circ$" and a constant "0". Then, $(H, \circ, 0)$ is called a *quasi hyper BCK-algebra* if satisfies the following axioms:
(H1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(H2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(H3) $x \circ H \ll x$.
and a quasi hyper BCK-algebra is called a *hyper BCK-algebra*, if
(H4) $x \ll y$ and $y \ll x$ imply $x = y$,
for all x, y, z in H, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A, \exists b \in B$ such that $a \ll b$. Nontrivial hyper quasi hyper BCK-algebra means that the hyperoperation " $\circ$ " is not singleton.

3. Some Results on Quasi BCK-algebras and Quasi Hyper BCK-algebras

In this section, we get some results that we need in the next section. Specially, we construct a BCK-algebra and a quasi hyper BCK-algebra from a nonempty set. First, we do some notations.

Definition 3.1. Let $(X, \circ)$ be a hyper BCK-algebra and R be an equivalence relation on X. If A and B are nonempty subsets of X, then
(i) $A\overline{\overline{R}}B$ means that for all $a \in A$, and $b \in B$, we have aRb.
(ii) R is called strongly regular on the right (on the left) if for all x of X, from aRb, it follows that $(a \circ x)\overline{\overline{R}}(b \circ x)$ $((x \circ a)\overline{\overline{R}}(x \circ b)$ respectively).
(iii) R is called strongly regular on the right (on the left) if for all x of X, from aRb, it follows that $(a \circ x)\overline{\overline{R}}(b \circ x)$ $((x \circ a)\overline{\overline{R}}(x \circ b)$ respectively).

Definition 3.2. Let X be a set with a binary operation " $*$ " and a constant "0". Then, $(X, *, 0)$ is called a *quasi BCK-algebra* if it satisfies the following conditions:
(QBCI-1) $((x * y) * (x * z)) * (z * y) = 0$,
(QBCI-2) $(x * y) * z = (x * z) * y$,
(QBCI-3) $(x * y) * x = 0$.

Theorem 3.3. *Every hyper BCK-algebra is a quasi hyper BCK-algebra.*

Definition 3.4. Let $(X, \circ)$ be a quasi hyper BCK-algebra. Then we set:
$\beta_1 = \{(x, x) \mid x \text{ in X}\}$ and, for every integer $n \geq 1$, β_n is the relation defined as follows:

$$x\beta_n y \Longleftrightarrow \exists (a_1, a_2, \dots, a_n) \in X^n,\ \exists u \in \mathcal{L}(a_1, a_2, \dots, a_n) : \{x, y\} \subseteq u$$

Obviously, for every $n \geq 1$, the relations β_n are symmetric, and the relation $\beta = \bigcup_{n\geq 1} \beta_n$ is reflexive and symmetric. Let β^* be the *transitive closure* of β. Then in the following theorem we show that β^* is strongly regular.

Theorem 3.5. *Let $(X, \circ)$ be a quasi hyper BCK-algebra. Then β^* is a strongly regular on X.*

Now, we define the product "$\overline{\circ}$" on $\frac{X}{\beta^*}$ in the usual manner:

$$\beta^*(x)\overline{\circ}\beta^*(y) = \{\beta^*(z) \mid z \in x\circ y\}$$

for all $x, y \in X$. In the following theorem we show that $(\frac{X}{\beta^*}, \overline{\circ})$ is a quasi BCK-algebra.

Theorem 3.6. *Let $(X, \circ)$ be a quasi hyper BCK-algebra. Then $(\frac{X}{\beta^*}, \overline{\circ})$ is a quasi BCK-algebra.*

Proof: By Theorem 3.5, β^* is strongly regular equivalence relation, then for any $x, y \in X, \beta^*(x)\overline{\circ}\beta^*(x)$ is singleton and so, $\beta^*(x)\overline{\circ}\beta^*(x) = \beta^*(z)$ for all $z \in \beta^*(x)\circ\beta^*(y)$. Clearly $(\frac{X}{\beta^*}, \overline{\circ})$ is a quasi BCK-algebra.

Theorem 3.7. *Let $(X, \circ)$ be a quasi hyper BCK-algebra. Then β^* is the smallest strongly regular equivalence relation on X, such that $\frac{X}{\beta^*}$ is a quasi BCK-algebra.*

Theorem 3.8. *Let X' be a nonempty set and $X = X' \cup \{0\}$. Then, there exists a binary hyperoperation " $\circ$ " on X such that $(X, \circ, 0)$ is a quasi hyper BCK-algebras.*
Proof: *Let $a \in X'$. Then for any $x, y \in X$, we define the binary hyperoperation " $\circ$ " on X as follows:*

$$x \circ y = \begin{cases} \{0, a\} & , \textit{if } x \in \{0, a\} \textit{ or } x = y \\ x & , \textit{otherwise} \end{cases}$$

Moreover, for any $x, y \in X$, we define $x \ll y$ by $0 \in x \circ y$. Clearly for any $x \in X$, $0 \in x \circ x$, $0 \in 0 \circ x$ and $0 \in a \circ x$. Then, $x \ll x$, $0 \ll x$ and $a \ll x$. It is easy to see that $(X, \circ, 0)$ is a quasi hyper BCK-algebra.

Theorem 3.9. *Let X' and Y' be two nonempty sets, $X = X' \cup \{0\}, Y = Y' \cup \{0\}$ and $|X'| = |Y'|$. Then, there exists a binary hyperoperation " $\circ$ " on X and Y, such that $(X, \circ, 0)$ and $(Y, \circ, 0)$ are two isomorphic quasi hyper BCK-algebras.*
Proof: *Let $0 \neq a \in X$. By Theorem 3.8, there exists a binary hyperoperation " $\circ$ " on X and Y, such that $(X, \circ, 0)$ and $(Y, \circ, 0)$ are quasi hyper BCK-algebras. Now, since $|X'| = |Y'|$, then there exists a bijection $\psi : X' \to Y'$. Let $\varphi : X \to Y$ is defined by $\varphi(x') = \psi(x')$, for any $x' \in X'$ and $\varphi(0) = 0$. It is easy to see that φ is an isomorphism.*

Theorem 3.10. *Let X' and Y' be two nonempty sets, $X = X' \cup \{0\}, Y = Y' \cup \{0\}$ and $|X'| = |Y'|$. Then, there exists a binary hyperoperation " $\circ$ " on X and Y, such that $(\frac{(X,\circ)}{\beta^*}, \overline{\circ}) \cong (\frac{(Y,\circ)}{\beta^*}, \overline{\circ})$.*

Proof: *Let $0 \neq a \in X$. By Theorem 3.9, there exists a binary hyperoperation " $\circ$ " and an isomorphism $f : (X, \circ, 0) \longrightarrow (Y, \circ, 0)$ such that $f(0) = 0$.Cleary, $\beta^*(0) = \{0, a\}$ and for any $x \in X$ such that $x \notin \beta^*(0)$, we have $\beta^*(x) = \{x\}$. Now, we define $\varphi : (\frac{(X,\circ,0)}{\beta^*}, \overline{\circ}) \to (\frac{(Y,\circ,0)}{\beta^*}, \overline{\circ})$ by $\varphi(\beta^*(x)) = \beta^*(f(x))$. It is easy to see that φ is well-defined and one to one.*

4. Fundamental Quasi BCK-algebra

In this section, by using the notion of fundamental relation which is define in the previous section, we define the concept of fundamental quasi BCK-algebra and we prove that any countable quasi BCK-algebra is a fundamental quasi BCK-algebra.

Definition 4.1. A quasi BCK-algebra $(X,*)$, is called a fundamental quasi BCK-algebra, if there exists a nontrivial quasi hyper BCK-algebra $(H,\circ)$, such that $(\frac{(H,\circ)}{\beta^*},\overline{\circ})\cong(X,*)$.

Theorem 4.2. *$\mathbb{N}$ is a fundamental quasi BCK-algebra.*

Corollary 4.3. *$\mathbb{N}_k$ is a fundamental quasi BCK-algebra, where $\mathbb{N}_k=\{1,2,\ldots,k\}$.*

Corollary 4.4. *Every countable set is a fundamental quasi BCK-algebra.*

Theorem 4.5. *Let $(X,*,0)$ be any finite BCK-algebra. Then, for any hyperoperation " $\circ$ " on X, such that $(X,\circ,0)$ is a quasi hyper BCK-algebra, there is not any isomorphic between $(X,*)$ and $(\frac{(X,\circ)}{\beta^*},\overline{\circ})$, that is $(X,\circ,0)\ncong(\frac{(X,\circ,0)}{\beta^*},\overline{\circ})$.*

Theorem 4.6. *$\mathbb{W}$ is fundamental quasi BCK-algebra of itself.*

Proof: *For any $n\in\mathbb{W}$, we set $A_{2n}=\{4n\}$ and $A_{2n+1}=\{2n+1,4n+2\}$. We define the hyperoperation " $\circ$ " on $\mathbb{W}$, for any $(x,y)\in A_i\times A_j$ by, $x\circ y=A_{i*j}$. First, we show that " $\circ$ " is well defined. It is easy to see that the family $\{A_{2n},A_{2n+1}\}_{n\in\mathbb{W}}$ is a partition for $\mathbb{W}$. Now, for any $(x,y)\in A_i\times A_j$, we define,*

$$x\ll y \quad \textit{if and only if } 0\in x\circ y$$

Since, $0\in x\circ y$ if and only if $0\in A_{i*j}$ if and only if $i*j=0$ if and only if $i\le j$, Then, for any $(x,y)\in A_i\times A_j$,

$$x\ll y \quad \textit{if and only if } i\le j \tag{1}$$

Hence, for any $n\in\mathbb{W}$, $A_0\ll A_n$, $A_n\ll A_n$ and $A_{2n}\ll A_{2n+1}$.It is easy to see that $(\mathbb{W},\circ,0)$ is a quasi hyper BCK-algebra.

Remark 4.7. In Theorem 4.6, we saw that the family $\{A_{2n}=\{4n\},A_{2n+1}=\{2n+1,4n+2\}\mid n\in\mathbb{W}\}$ is a partition of $\mathbb{W}$. Now, if $\varphi:\mathbb{W}\longrightarrow X$ be a bijection, then, the family $\{\varphi(A_{2n})=\{\varphi(4n)\},\varphi(A_{2n+1})=\{\varphi(2n+1),\varphi(4n+2)\}\mid n\in\mathbb{W}\}$ is a partition of X, too.

Theorem 4.8. *Let X' be a infinite countable set and $X=X'\cup\{0\}$. Then, there exists an operation " $*$ " and a hyperoperation " $\circ$ " on X, such that $(\frac{(X,\circ,0)}{\beta^*},\overline{\circ})\cong(X,*,0)$. That is X is a fundamental quasi BCK-algebra of itself.*

Proof: *Since $|X'|=|\mathbb{W}|$, there exists a bijection $\psi:\mathbb{W}\to X'$. Let $\varphi:\mathbb{W}\to X$ is define by $\varphi(x')=\psi(x')$, for any $x'\in\mathbb{W}$ and $\varphi(0)=0$. For any $x,y\in X$, we define hyperoperation " $\circ$ " on X, by*

$$x\circ y=A_{\varphi(i)*\varphi(j)}$$

where $(x,y)\in A_{\varphi(i)}\times A_{\varphi(j)}$, for $i,j\in\mathbb{W}$. Moreover, we let

$$x\ll y \textit{ if and only if } 0\in x\circ y$$

Since $0\in x\circ y$ if and only if $0\in A_{\varphi(i)*\varphi(j)}$ if and only if $A_{\varphi(i)*\varphi(j)}=A_{\varphi(0)}$ if and only if $\varphi(i)*\varphi(j)=\varphi(0)=0$ if and only if $\varphi(i)\le\varphi(j)$, hence, for any $(x,y)\in A_{\varphi(i)}\times A_{\varphi(j)}$,

$$x\ll y \textit{ if and only if } \varphi(i)\le\varphi(j) \tag{2}$$

and so for any $x \in X$, $A_{\varphi(0)} \ll A_x$, $A_x \ll A_x$. *It is easy to see that* $(X, \circ, 0)$ *is a quasi hyper* BCK*-algebra. Now, we define the map* $\phi : (\frac{(X,\bullet,0)}{\beta^*}, \overline{\circ}) \longrightarrow (X, *, 0)$ *by,*

$$\phi(\beta^*(x)) = \phi(\beta^*(\varphi(n))) = \begin{cases} \varphi(n) & \textit{if } n \textit{ is odd} \\ \varphi(\frac{n}{2}) & \textit{if } n \textit{ is even} \end{cases}$$

where $x = \varphi(n)$, *for* $n \in \mathbb{W}$. *Clearly* φ *is an isomorphism and so* $(\frac{(X,\circ,0)}{\beta^*}, \overline{\circ}) \cong (X, *, 0)$. *Hence* X *is a fundamental quasi* BCK*-algebra of itself. .*

Open Problem: If $(X, *)$ be an infinite non-countable quasi BCK-algebra, then is it X as a fundamental quasi BCK-algebra of itself?

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MACAULAY'S SELF-DUALITY THEOREM IN ANY DIMENSION

S. H. HASSANZADEH

ABSTRACT. We prove that an infinity term of a Čech-Kozul spectral sequence is a (uniform) self-dual submodule of $H^0_{\mathfrak{m}}(R)$.

This is a joint work with M. Chardin, J.P. Jouanolou and A. Simis.

1. INTRODUCTION

Let f be a homogeneous polynomial in a homogeneous polynomial ring over a field k, $P = k[x_0, \cdots, x_r]$. Let $X := V(f) \subseteq \mathbb{P}^r$ be the hypersurface defined by f, $J = (\partial f/\partial x_0, \cdots, \partial f/\partial x_0)$ and $R = R(f) = P/J$. A theorem of Macaulay in the case where X is non-singular states the self-duality of the Artinian Gorenstein ring $R(f)$ [V, Theorem 6.19]. Applications of this self-duality involves the Torelli theorem and the Symmetriser lemma which in its own turn provides the determination of the Hodge loci for families of hypersurfaces and the triviality of Abel-Jacobi map e.g.[V, 6.3,7.3]. The immediate next question is about the hypersurfaces with higher dimensional singularities. May we still have Torelli's type theorems or determination of Hodge decomposition in higher dimensions? Since all of these applications are based on the Macaulay's duality theorem, we have to know the style of the self-duality in higher dimensions.

So far, in the case where X has singularities of higher dimension nothing is known about the Macaulay's self-duality theorem (or at least we are not aware of). In this work we present a module, call it $\mathbf{K}$, which seems to be the true Macaulay's self-dual object in any dimension.

A careful study of the proofs of the self-duality in dimension $0, 1$ such as [HS],[C],[Se] and [V] shows that in these cases Macaulay's self-dual object is $H^0_{\mathfrak{m}}(R)$ which is an infinity term of a Čech-Kozul spectral sequence- namely it is ${}^{\infty}E^{0,0}_{\text{hor}}$. However in higher dimensions this infinity term is only a submodule of $H^0_{\mathfrak{m}}(R)$ and not necessarily the whole finite part. Our conjecture is: **conjecture.** The R-module $\mathbf{K} := {}^{\infty}E^{0,0}_{\text{hor}}$ is a

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(uniform) self-dual submodule of $H^0_{\mathfrak{m}}(R)$.

We prove the conjecture for monomial ideals. In the case where $\dim(R) = 2$, it is shown that $\mathbf{K}$ is piece-wise self-dual while in the cases $\dim(R) = 0, 1$ it is uniformly self-dual.

One of the techniques to prove the duality is via a special map $\mathfrak{J}$ already introduced by Jouanolou in [J-Resultant] and [J-Explicit], some applications and particular cases are studied as well.

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SOME RESULTS ON SELF-COGENERATOR MODULES

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Abstract. In this talk, we investigate some properties of self-cogenerator modules. The aim of this talk is to present some applications. Let M be a right module. If M is self-cogenerator Baer, we show that M is dual Baer.

This is a joint work with Yahya Talebi‡.

1. Introduction

Throughout this talk, R is an associative ring with identity, modules are right and unitary over it and $S = End_R(M)$ is the ring of R-endomorphisms of M. Clearly, the module M is a left S and right R-bimodule. $Rad(M)$, $Soc(M)$ will indicate Jacobson radical of M, Socle of M.

A submodule K of M is denoted by $K \leq M$. A submodule K of M is called *essential* in M (denoted by $K \subseteq^e M$), if $K \cap L \neq 0$ for every nonzero submodule L of M, and a submodule K of M is called *small* in M (denoted by $K \ll M$), if $N + K \neq M$ for any proper submodule N of M. A nonzero module is said to be *uniform* if each nonzero submodule is essential. It is said to be *hollow* if each proper submodule is small. Recall that the singular submodule $Z(M)$ of a module M is the set of $m \in M$ such that $mI = 0$ for some essential right ideal I of R. In [3], Johnson introduced the singular submodule of a module. If $Z(M) = 0$ ($Z(M) = M$), then M is called a *nonsingular* (*singular*) module.

A ring R (module M) is called a right *dual* ring (right dual module) if every right ideal I of R is an annihilator, that is, $r_R l_R(I) = I$ ($r_M l_S(N) = N$ for each submodule N). By [2, Lemma 24.4], for each submodule N of M we have $r_M l_S(N)/N = Rej_{M/N}(M) = \bigcap\{kerh \mid h \in Hom(M/N, M)\}$. Thus M is a dual module if and only if $Rej_{M/N}(M) = 0$ for each submodule N of M, if and only if M/N is cogenerated by M for each submodule N of M. Recall that a module M is said to be a self-cogenerator if it cogenerates each of its factors, that is, $N = r_M l_S(N)$ for all submodules N of M [8]. So self-cogenerator and dual module is the same. Clearly semisimple and cogenerator modules are self-cogenerator.

Let $S = End_R(M)$ be a ring and let ${}_SM$ be a left S-module. Then for any $X \subseteq M$ and $Y \subseteq S$, the left annihilator of X in S and the right annihilator of Y in M are

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$l_S(X) = \{s \in S \mid sx = 0$ for all $x \in X\}$ and $r_M(Y) = \{m \in M \mid ym = 0$ for all $y \in Y\}$, respectively.

2. Properties of self-cogenerator modules

Recall that an R-module M is called coretractable if, for any proper submodule K of M, there exists a nonzero homomorphism $f : M \longrightarrow M$ with $f(K) = 0$, that is, $Hom_R(M/K, M) \neq 0$ (see [1]). Recall that an R-module M is cosemisimple if for any $K \leq M$, $Rad(M/K) = 0$ (see [8]).

Proposition 2.1. *Any self-cogenerator module is coretractable.*

A ring R is said to be right Kasch if every simple right R-module can be embedded into R_R. By [1, Theorem 2.14], if R is a right Kasch ring, then R_R is a coretractable module. But it is clear that R_R is not self-cogenerator.

Proposition 2.2. *Let M be coretractable. If M is cosemisimple, then M is a self-cogenerator module.*

We have the following implications:
cogenerator module$\Longrightarrow$ self-cogenerator module$\Longrightarrow$ coretractable module

Proposition 2.3. *Let M be a self-cogenerator module. Then:*
(1) *If $l_S(N) \subseteq^e {}_SS$, then $N \ll M$.*
(2) *If $I \subseteq^e {}_SS$, then $r_M(I) \ll M$.*
(3) *If $N \ll M$, then $l_S(N) \subseteq^e {}_SS$.*
(4) *$Soc({}_SS) \subseteq l_S(Rad(M))$.*
(5) *$Z({}_SM) = Rad(M)$.*

Corollary 2.4. *Let M be a self-cogenerator R-module and $\nabla(M) = \{f \in S \mid f(M) \ll M\}$. Then $Z({}_SS) = \nabla(M)$.*

Following [8, p. 261], an R-module M is called *semi-injective* if for any $f \in S$, $Sf = l_S(ker(f)) = l_S(r_M(f))$.

Proposition 2.5. *Let M be a semi-injective self-cogenerator module and N a proper submodule of M. If M/N is hollow, then $l_S(N)$ is a uniform ideal of S.*

Consider the following properties for an R-module M:
(C_1) Every submodule of M is essential in a direct summand of M.
(C_2) Every submodule isomorphic to a direct summand of M is also a direct summand.
(C_3) If M_1 and M_2 are direct summands of M with $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a direct summand of M.

An R-module M is called *continuous* if it has (C_1) and (C_2), M is called *quasi-continuous* if it has properties (C_1) and (C_3), and M is called an *extending* if it has property (C_1).

A ring R is called left PP-ring, if every cyclic left ideal is projective, equivalently, if the left annihilator of each element of R is a direct summand of ${}_RR$.

Theorem 2.6. *Let M be self-cogenerator and S a PP-ring. Then M is continuous.*

Corollary 2.7. *Let M be a self-cogenerator module and S a PP-ring, $\Delta(M) = \{f \in S \mid Kerf \subseteq^e M\}$. Then $J(S) = \Delta(M)$.*

Theorem 2.8. *Let R be a commutative ring and M an R-module. If for any submodule N of M, there exists a two sided ideal I of R such that $N = r_M(I)$, then M is a self-cogenerator R-module.*

Corollary 2.9. *Let R be a commutative domain and M be an R-module. If for any submodule N of M, there exists a two sided ideal I of R such that $N = r_M(I)$, then M is hollow.*

3. Applications

In this section, we consider some applications of self-cogenerator and self-generator modules in other modules, in particular in Baer, dual Baer and extending modules. This is the focus of our investigations in this paper. We provide some additional motivation as follows. In [4], Kaplansky introduced the concept of a Baer ring. A ring R is called Baer if the right annihilator of any nonempty subset of R is generated by an idempotent. According to [7], an S-module M is called a Baer module if the right annihilator in M of any left ideal of S is a direct summand of M. In [5], a module M is called a dual Baer module if for every right ideal I of S, $\sum_{\phi \in I} Im\phi$ is a direct summand of M.

Lemma 3.1. *(*[9, Lemma 1]*) If $I \leq S$ and $N \leq M$, then:*
(1) $D_S(N)M \subseteq N$.
(2) $N \subseteq r_M(l_S(N))$.
(3) $l_S(N)D_S(N) = 0$.
(4) $D_S(r_M(I)) = r_S(I)$.
(5) $l_S(IM) = l_S(I)$.

Lemma 3.2. *(*[9, Lemma 2]*) Let M be an R-module. Then:*
(1) *If $D_S(N)M = N$, then $l_S(N) = l_S(D_S(N))$.*
(2) *If $r_M(l_S(N)) = N$, then $D_S(N) = r_S(l_S(N))$.*

Recall from [8], a module M is said to be a self-generator if it generates each of its submodules, that is, $N = Hom(M, N).M$ for all $N \leq M$. We call M *weakly self-generator*, if M generates $r_M(I)$ for every $I \leq {}_S S$.

Proposition 3.3. *The following are equivalent for an S-module M:*
(1) *M is a Baer module;*
(2) *M is weakly self-generator and S is a Baer ring.*

Corollary 3.4. *Let M be weakly self-generator. If M is a dual Baer module, then M is Baer.*

Let M be a module. We call M *weakly self-cogenerator*, if $r_M l_S(I) = IM$ for each $I \leq {}_S S$ and $IM \lneq M$.

Every self-cogenerator module is weakly self-cogenerator

Proposition 3.5. *The following are equivalent for a module M:*
(1) *M is dual Baer module;*
(2) *M is weakly self-cogenerator and S is a Baer ring.*

Example 3.6. (1) *If G is a divisible abelian group, then G is dual Baer and so $End(G)$ is a Baer ring.*

(2) *Let M be the $\mathbb{Z}$-module $\mathbb{Z}_{\mathbb{Z}}$. Then $End(M) \cong \mathbb{Z}$ is a Baer ring but M is not a dual Baer module.*

Proposition 3.7. *Let M be a weakly self-cogenerator module. If M is a Baer module, then M is dual Baer.*

Corollary 3.8. *Let M be a self-cogenerator module. If M is a Baer module, then M is dual Baer.*

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CUBIC SEMISYMMETRIC GRAPHS OF ORDERS $38p^2$ AND $46p^2$

A. IMANI†

Abstract. A graph X is said to be G-semisymmetric if it is regular and there exists a subgroup G of $A := Aut(X)$ acting transitively on its edge set but not on its vertex set. In the case of $G = A$, we call X a semisymmetric graph. Let p be a prime. It was shown by Folkman (J. Combin. Theory 3(1967)215 − 232) that a regular edge-transitive graph of order $2p$ or $2p^2$ is necessarily vertex-transitive. The smallest semisymmetric graph is Folkman graph. In this talk, we prove that for every prime p, there is no connected cubic semisymmetric graph of order $38p^2$ or $46p^2$.

This is a joint work with A.A. Talebi‡ and N. Mehdipoor††.

1. Introduction

All graphs considered in this paper are assumed to be undirected, finite, simple and connected unless stated otherwise. For a graph X, we use $V(X)$, $E(X)$, $Arc(X)$ and $Aut(X)$ to denote its vertex set, edge set, arc set and full automorphism group, respectively. Let G be a subgroup of $Aut(X)$. For $u, v \in \mathrm{V}(X)$, denote by uv the edge incident to u and v in X, and by $N_X(u)$ the neighborhood of u in X, that is, the set of vertices adjacent to u in X. A graph $\widetilde{X}$ is called a covering of a graph X with projection $p : \widetilde{X} \to X$ if there is surjection $p : V(\widetilde{X}) \to V(X)$ such that $p|_{N_{\widetilde{X}}(\widetilde{v})} : N_{\widetilde{X}}(\widetilde{v}) \to N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\widetilde{v} \in p^{-1}(v)$. A permutation group G on a set Ω is said to be semiregular if the stabilizer G_v of v in G is trivial for each $v \in \Omega$, and it is said to be regular if G is transitive and semiregular. Let K be a subgroup of $Aut(X)$ such that K is intransitive on $V(X)$. The quotient graph X/K induced by K is defined as the graph such that the set Ω of K-orbits in $V(X)$ is the vertex set of X/K and $B, C \in \Omega$ are adjacent if and only if there exist $u \in B$ and $v \in C$ such that $\{u, v\}$ $\in E(X)$. A covering $\widetilde{X}$ of X with a projection p is said to be regular (or N-covering) if there is a semiregular subgroup N of the automorphism group $Aut(\widetilde{X})$ such that graph X is isomorphic to the quotient graph $\widetilde{X}/N$, say by h, and the quotient map $\widetilde{X}$ $\to \widetilde{X}/N$ is the composition ph of p and h (in this paper, all functions are composed from left to right). If N is cyclic or elementary Abelian then $\widetilde{X}$ is called a cyclic or an elementary Abelian covering of X, and if $\widetilde{X}$ is connected N becomes the covering transformation group. For a graph X and a subgroup G of $Aut(X)$, X is said to be G-vertex-transitive, G-edge-transitive or G-s-arc-transitive if G is transitive on the

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sets of vertices, edges or s-arcs of X respectively, and G-s-regular if G acts regularly on the set of s-arcs of X. Similarly, a graph is G-semisymmetric if it is G-edge-transitive but not G-vertex-transitive. A graph X is said to be vertex-transitive, edge-transitive, s-arc-transitive or s-regular if X is $Aut(X)$-vertex-transitive, $Aut(X)$-edge-transitive, $Aut(X)$-s-arc-transitive or $Aut(X)$-s-regular respectively. In particular, 1-arc-transitive means arc-transitive or symmetric. It can be shown that a G-edge-transitive but not G-vertex-transitive graph is necessarily bipartite, where the two partite parts of the graph are orbits of G. Moreover, if X is regular these two partite sets have equal cardinality. Covering techniques have long been known as a powerful tool in topology and graph theory. Regular covering of a graph is currently an active topic in algebraic graph theory. The class of semisymmetric graphs was first introduced by Folkman [10], where several infinite families of such graphs were constructed and eight open problems were posed which spurred the interest in this topic Subsequently, Bouwer [1, 2], Titov, Klin, Iofinova and Ivanov, Ivanov, Du and Xu and others did much work on semisymmetric graphs. They gave new constructions of such graphs by combinatorial and group-theoretical methods. The answers to most of Folkman's open problems are now known. Note that a semisymmetric graph cannot be a covering of the complete graph K_4 of order 4 because K_4 is not bipartite. A simple observation then shows that there is no connected cubic semisymmetric graph of order $4p$, $4p^2$, or $4p^3$. There is no connected cubic semisymmetric graph of order $6p$ (see[8]), and Lu et al. [16] classified connected cubic semisymmetric graphs of order $6p^2$. Malnič et al.[18] classified cubic semisymmetric graph of order $2p^3$ for a prime p, while Folkman [10] proved that there is no cubic semisymmetric graphs of order $2p$ or $2p^2$. Some general methods of elementary Abelian coverings were developed in [5, 17]. The s-regular cyclic coverings and elementary Abelian coverings of the three-dimensional hypercube Q_3 were classified in [8, 9].

2. Cubic Semisymmetric graphs of orders $38p^2$ or $46p^2$

We start with introducing four proposition for later applications in this paper. The first proposition is a special case of [16, Lemma 3.2].

Proposition 2.1. *Let X be a connected cubic semisymmetric graph with bipartition sets $U(X)$ and $W(X)$. Moreover, suppose that N is a normal subgroup of $A := Aut(X)$. If N is intransitive on bipartition sets, then N acts semiregularly on both $U(X)$ and $W(X)$, and X is an N-regular covering of an A/N-semisymmetric graph.*

Proposition 2.2. [11] *The vertex stabilizers of a connected cubic G- semisymmetric graph X ($G \leq Aut(X)$) have order $2^r.3$ where $0 \leq r \leq 7$.*

Proposition 2.3. [13] *i) Let p and q be primes and let a and b be non negative integers.Then every group of order $p^a p^b$ is solvable.*
ii) Every finite group of odd order is solvable.

Proposition 2.4. [22] *Let G be a finite group. If G has an abelian Sylow p-subgroup, then p does not divide $|G' \cap Z(G)|$.*

Now, by using the covering technique and group-theoretical construction, we investigate cubic semisymmetric graphs of orders $38p^2$, $46p^2$, where p is a prime.

Lemma 2.5. *Let X be a connected cubic semisymmetric graph of the orders $38p^2$, $46p^2$, with bipartition sets $U(X)$ and $W(X)$, where p is a prime, and $n \geq 2$. Moreover, suppose that N is a minimal normal subgroup of $A := Aut(X)$. Then, N is solvable, elementary Abelian, and intransitive on the bipartition sets.*

Theorem 2.6. *There is no connected cubic semisymmetric graph of order $38p^2$, where p be a prime.*

Theorem 2.7. *Let p be a prime. Then there is no connected cubic semisymmetric graph of order $46p^2$.*

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X-COVERS IN THE CATEGORY $S-$ act AND ITS FULL SUBCATEGORY $C^{-1}S-$ act

S. IRANNEZHAD

Abstract. In (Proc. Edinb. Math. Soc. **57**: 589–617, 2014) Bailey and Renshaw studied **X**-covers of acts over monoids where **X** is a class of S-acts. Semigroups of fractions have been studied, mostly following the analogy with localization, by Maury (C. R. Acad. Sci. Paris **247**: 254–255, 1958), Bouvier (Séminaire P. Lefebvre, 1968/69, no.IO. Dép. Math., Univ. Claude Bernard, Villeurbanne 1969) et al. Preservation and reflection of certain properties by functors is a very useful information in the investigation of categories and functors. It is a natural question whether the functor C^{-1} preserves and reflects **X**-covers. In this note we give a negative answer to the first case and a positive answer to the second case. We also give conditions under which C^{-1} preserves **X**-covers.

This is a joint work with A. Madanshekaf.

1. Introduction

In 2008, Mahmoudi and Renshaw [4] initiated a study of flat covers of acts over monoids. Their definition of cover concerned with coessential epimorphisms. Bailey and Renshaw introduced the concept of an **X**-cover and **X**-precover for a class of S-acts **X** in [2]. Their definition is the analogue of Enoch's definition for covers of modules over rings.

In the this talk we establish Enoch's notion of cover to the category of acts over monoids and focus on the behavior of the functor C^{-1}, with respect to **X**-covers.

Let S be any commutative monoid and let C be a multiplicative subset of S; that is, $1 \in C$ and C is closed under multiplication. The monoid of fractions $C^{-1}S$ consists of all fractions s/a with $s \in S$ and $a \in C$, with $s/a = t/b$ if and only if $cbs = cat$ for some $c \in C$ and multiplication $(s/a)(t/b) = st/ab$.

Let C be a submonoid of S and X be an S-act. Then the localization of X with respect to C, denoted by $C^{-1}X$, is defined as for $C^{-1}S$. $C^{-1}X$ is a $C^{-1}S$-act via $(s/a) \cdot (x/b) = sx/ab$. It is known [5] that C^{-1} is given by

$$C^{-1} : S - \mathbf{Act} \longrightarrow C^{-1}S - \mathbf{Act}$$

$$\begin{array}{ccc} X & \rightsquigarrow & C^{-1}X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle C^{-1}f} \\ Y & \rightsquigarrow & C^{-1}Y, \end{array}$$

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where $C^{-1}f(x/a) = f(x)/a$ for $x/a \in C^{-1}X$, is a covariant functor. We refer the reader to [5] for more details.

2. BEHAVIOR OF THE FUNCTOR C^{-1}

Recall that an S-act ${}_SA$ is

- *free* if it has a basis.
- *injective* if the functor $\hom_S(-, A)$ takes monomorphisms to epimorphisms.
- *projective* if the functor $\hom_S(A, -)$ preserves epimorphisms.
- said to satisfies *condition* (BF) if for all $a, a' \in A$, $s, s', t, t' \in S$ such that $sa = s'a'$ and $ta = t'a'$, there exist $a'' \in A$, $u, v \in S$, such that $a = ua''$, $a' = va''$, $su = s'v$, $tu = t'v$.
- said to satisfies *condition* (P) if for all $a, a' \in A$, $s, s' \in S$ such that $sa = s'a'$, there exist $a'' \in A$, $u, v \in S$ with $a = ua''$, $a' = va''$ and $su = s'v$.
- *flat* if the functor $- \otimes {}_SA$ preserves monomorphisms.
- *weakly flat* if the functor $- \otimes {}_SA$ preserves all embeddings of right ideals into S.
- *principally weakly flat* if the functor $- \otimes {}_SA$ preserves all embeddings of principal right ideals into S.
- *torsion free* if for any $x, y \in A$ and for any left cancellable element $c \in S$ the equality $cx = cy$ implies $x = y$.

Let S be a monoid and A be an S-act. Unless otherwise stated, in the rest of this paper, $\mathbf{X}$ will be a class of S-acts closed under isomorphisms. By an $\mathbf{X}$-*precover* of A we mean an S-morphism $g : X \to A$ for some $X \in \mathbf{X}$ such that for every S-morphism $g' : X' \to A$, with $X' \in \mathbf{X}$, there exists an S-morphism $f : X' \to X$ with $g' = gf$. If in addition the precover satisfies the condition that each S-morphism $f : X \to X$ with $gf = g$ is an isomorphism, then we shall call it an $\mathbf{X}$-*cover*.
Recall from [5] that if S is a commutative monoid, C is a submonoid of S, and X is an S-act, on which C acts injectively, every congruence ρ on X induces a congruence $\hat{\rho}$ on $C^{-1}X$, in which $f\hat{\rho}g$ if and only if $cf = x/1$, $cg = y/1$ and $x\rho y$, for some $c \in C$ and $x, y \in X$.

In what follows, our main results are given.

Theorem 2.1. *Let $C \subset S$ be a multiplicative set. Then*

1. *If $f : A \to B$ is an S-act homomorphism, then $C^{-1}f : C^{-1}A \to C^{-1}B$ defined by $(C^{-1}f)(x/a) = f(x)/a$, $x/a \in C^{-1}A$ is a $C^{-1}S$-act homomorphism.*
2. *If $f : A \to B$ is an S-act isomorphism, then $C^{-1}f$ is a $C^{-1}S$-act isomorphism.*
3. *If B is an S-act, on which C acts injectively, then $C^{-1}(B/\rho) \cong C^{-1}(B)/(\hat{\rho})$, where ρ is an S-act congruence on B.*
4. *If A is an S-act, then $C^{-1}S_S \otimes {}_SA \cong C^{-1}A$.*
5. *$C^{-1}S$ is a flat S-act.*

Recall that the disjoint union of a family $(A_i)_{i\in I}$ of S-acts is denoted by $\dot{\bigcup}_{i\in I}A_i$ and is actually the coproduct of A_i's in $S-\mathbf{Act}$.

Theorem 2.2. *Let $C \subset S$ be a multiplicative set. If $(A_i)_{i\in I}$ is a family of S-acts, then there is a natural $C^{-1}S$-act isomorphism $C^{-1}(\dot{\bigcup}_{i\in I}A_i) \to \dot{\bigcup}_{i\in I}(C^{-1}A_i)$.*

Theorem 2.3. *Let $C \subset S$ be a multiplicative set. Then*

1. *If A is a free S-act, then $C^{-1}A$ is a free $C^{-1}S$-act.*
2. *If A is a projective S-act, then $C^{-1}A$ is a projective $C^{-1}S$-act.*
3. *Let A be an $C^{-1}S$-act which is injective as an S-act, then A is an injective $C^{-1}S$-act.*
4. *If A is a flat S-act, then $C^{-1}A$ is a flat $C^{-1}S$-act.*

Theorem 2.4. *Let A be a $C^{-1}S$-act in the class $\mathbf{X}$. Then $A \in \mathbf{X}$ as an S-act, where $\mathbf{X}$ can stand for injectivity, strongly flatness, condition (P), flatness, weak flatness, principal weak flatness and torsion freeness.*

Theorem 2.5. *If $C \subset S$ is multiplicative and $X \to A$ is an flat-cover of $C^{-1}S$-acts, then $X \to A$ is also an flat-cover of S-acts.*

Lemma 2.6. *Let S be a commutative monoid with zero and let M be its unique maximal ideal and $C = S \setminus M$ the set of invertible elements of S, then for every S-act A, $C^{-1}A \cong A$ as S-acts.*

Theorem 2.7. *Let S be a commutative monoid with zero and let M be unique maximal ideal of S and $C = S \setminus M$ be the set of invertible elements of S. If $X \to A$ is an $\mathbf{X}$-cover of S-acts, then $C^{-1}X \to C^{-1}A$ is an $\mathbf{X}$-cover of $C^{-1}S$-acts, where $\mathbf{X}$ is a class of acts which is closed under isomorphisms.*

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ON RATIONAL GROUPS WHOSE IRREDUCIBLE CHARACTERS VANISH ONLY ON p-ELEMENTS

SAEID JAFARI[†]

Abstract. A finite group whose irreducible complex characters are rational valued is called a rational group. Our aim in this talk is to study the rational groups whose irreducible characters vanish only on p-elements for a prime number p.

This is a joint work with Hesam Sharifi[‡].

1. Introduction

Definition 1.1. *Let G be a finite group, all whose irreducible complex characters are rational valued. Such a group G is called a* ***rational group*** *or a* ***$\mathbb{Q}$-group***.

The importance of studying $\mathbb{Q}$-groups can be deduced, looking at the history of character theory. Some examples of $\mathbb{Q}$-groups are the symmetric groups S_n, Dihedral groups of orders 2, 4, 6, 8, 12, elementary abelian 2-groups and the Weyl groups of the complex Lie algebras. One can find some important properties of $\mathbb{Q}$-groups in [2].

Definition 1.2. *Let G be a finite group and χ be a complex character of G. If $\chi(g) = 0$ for some $g \in G$ then g is called a zero of χ and we say χ vanishes on g.*

In this context, by a character we always mean a complex character. A well-known theorem of Burnside asserts that every nonlinear irreducible character of a finite group H vanishes on some element $h \in H$. More precisely Malle, Navarro and Olsson in [3], have shown that if χ is a nonlinear irreducible character of H then there exists a p-element $h \in H$ such that $\chi(h) = 0$, where p is a prime number dividing the order of H.

Definition 1.3. *Let $H \subseteq F$, with $1 < H < F$. Assume that $H \cap H^x = 1$ whenever $x \in F \setminus H$. Then H is a* ***Frobenius complement*** *in F. A group which contains a Frobenius complement is called a* ***Frobenius group***.

Here we are interested on studying rational groups whose irreducible characters vanish only on p-elements for a fixed prime p. We claim that if a $\mathbb{Q}$-group G satisfies such property then $p = 2$ and G is solvable. Especially we will show, besides the finite elementary abelian 2-groups, every $\mathbb{Q}$-group whose irreducible characters vanish only on involutions, satisfies $G \cong Z(G) \times F$, where F is a Frobenius group with an elementary abelian 3-group as Frobenius kernel and $\mathbb{Z}_2$ as Frobenius complement.

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In this talk our notations are standard. Precisely we denote by $Irr(G)$, the set of irreducible characters of G and by $nl(G)$, the set of nonlinear irreducible characters of G . The notation $K : H$ will be used to denote semidirect product of K and H. Also $O_p(G)$ denotes the largest normal p-subgroup of G. By Q_8 we mean the quaternion group of order 8.

D. Bubboloni, S. Dolfi and P. Spiga in [1] have done the major part of the work. Here we are looking at the finite groups which are discussed there, in view of rationality. The authors of [1] have shown that besides the obvious cases of abelian groups and p-groups, for every finite group G, whose irreducible characters vanish only on p-elements, $G/Z(G)$ is a Frobenius group with a Frobenius complement of p-power order and $Z(G) = O_p(G)$. In the same reference, it is shown that there exists some Frobenius groups which do not satisfy the condition, vanishing only on p-elements for some prime p. Here this question arise, if we put the restriction of rationality on the Frobenius group G, then what can be said about the order of the vanishing elements of G? We try to answer this question, partially.

2. Results

Theorem 2.1. *(The main theorem of* [7]*) If G is a Frobenius $\mathbb{Q}$-group, then exactly one of the following occurs:*

(i) *$G \cong E(3^n) : \mathbb{Z}_2$, where $n \geq 1$ and $\mathbb{Z}_2$ acts on $E(3^n)$ by inverting every non-identity element.*

(ii) *$G \cong E(3^{2m}) : Q_8$, where $m \geq 1$ and $E(3^{2m})$ is a direct sum of m copies of the 2-dimensional irreducible representations of Q_8 over the field with 3 elements.*

(iii) *$G \cong E(5^2) : Q_8$, where $E(5^2)$ is the 2-dimensional irreducible representations of Q_8 over the field with 5 elements.*

The next lemma asserts that the only nonvanishing columns in the character table of a p-group are the columns related to the conjugacy classes containing in the center of group.

Lemma 2.2. *(*[1]*, Lemma 2.9) Let P be a p-group and $x \in P$. If $\chi(x) \neq 0$ for every $\chi \in Irr(P)$ then $x \in Z(P)$.*

Theorem 2.3. *Let G be a finite group all whose irreducible characters vanish only on involutions. Let P be a 2-Sylow subgroup of G. Then the following are true.*

(i) *If $a \in P \setminus Z(P)$ then $C_G(a) \subseteq P$.*

(ii) *If G is a $\mathbb{Q}$-group then $Z(G) = O_2(G)$, and if $Z(G) = 1$ then for every non-identity element $a \in Z(P)$ we have $C_G(a) = P$.*

Proof. As every nonlinear character of G vanishes only on 2-elements, by a consequence of the main theorem of [8], for every $\chi \in nl(G)$, $\chi(1)$ is an even integer and then using one of Thompson's theorems in [6], G has a normal 2-complement H.
Now suppose that $a \in P \setminus Z(P)$. Using theorem B of [5] there exists $\chi \in Irr(G/H)$ such that $\chi(a) = 0$. If x be a nonidentity element in $H \cap C_G(a)$ then $\chi(ax) = 0$. Thus by the hypothesis ax is an involution. So x is an involution, which violates the assumption $x \in H$. Therefore $C_G(a) \subseteq P$.

Now assume that G is a $\mathbb{Q}$-group. Then $Z(G)$ is an elementary abelian 2-group. So $Z(G) \subseteq O_2(G)$. Since $H \cap O_2(G) = 1$ and both H and $O_2(G)$ are normal subgroups

of G, we have $O_2(G) \subseteq C_G(H)$ and hence by part (i) we have $O_2(G) \subseteq Z(P)$. Now, as both H and P are contained in $C_G(O_2(G))$ we conclude $G \subseteq C_G(O_2(G))$. That is $O_2(G) = Z(G)$.
Now if $Z(G) = 1$, using [[4] Theorem 6.22], G is a relative M-group with respect to H, so one can deduce $C_G(a) \subseteq P$ and so $C_G(a) = P$. □

Theorem 2.4. *Let G be a nonabelian rational group in which every $\chi \in Irr(G)$ vanishes only on p-elements for some prime number p. Then $p = 2$ and $G/Z(G)$ is a Frobenius $\mathbb{Q}$-group.*

Proof. By a similar argument as in the beginning of the proof of Theorem 2.3, G has a normal p-complement, say H. Then by [2], G/H is a $\mathbb{Q}$-group and $|G/H|$ is even. So G/H is a 2-group, which implies $p = 2$. As $O_2(G/O_2(G))$ is trivial, by Theorem 2.3, $Z(G/Z(G)) = 1$ and so again by Theorem 2.3, $G/Z(G)$ is a Frobenius $\mathbb{Q}$-group. □

Theorem 2.5. *Let G be a nonabelian rational group all whose irreducible characters vanish only on involutions then $G \cong Z(G) \times F$, where F is a Frobenius group with an elementary abelian 3-group as Frobenius kernel and $\mathbb{Z}_2$ as Frobenius complement.*

Proof. By Theorem 2.4, $G \cong Z(G) \times F$, where F is a Frobenius group. Suppose that K is the Frobenius kernel and H is the Frobenius complement of F. Since all the characters induced from K vanish on H, by hypothesis, H is an elementary abelian 2-group. Therefore according to Theorem 2.1 the proof is complete. □

In order to discuss about the question which mentioned above we can say:

Theorem 2.6. *Let G be as in item (i) in Theorem 2.1. Then every irreducible character of G vanishes only on involutions.*

Proof. By Theorem 2.4 every irreducible character of G vanishes only on 2-elements; But every 2-element of G is an involution. □

Theorem 2.7. *Let G be as item (ii) for $m = 1$, or as item (iii) in Theorem 2.1. Then every irreducible character of G vanishes only on 2-elements.*

Proof. For these two special cases one can use the GAP software to find the character table of the groups and see that they satisfy the assertion. The left table is the character table of $G \cong E(3^2) : Q_8$ and the right one is for $G \cong E(5^2) : Q_8$.

2	3	2	2	3	.	2
3	2	.	.	.	2	.
	1a	4a	4b	2a	3a	4c
2P	1a	2a	2a	1a	3a	2a
3P	1a	4a	4b	2a	1a	4c
X.1	1	1	1	1	1	1
X.2	1	-1	-1	1	1	1
X.3	1	-1	1	1	1	-1
X.4	1	1	-1	1	1	-1
X.5	2	.	.	-2	2	.
X.6	8	.	.	.	-1	.

2	3	2	2	3	.	2	.	.
5	2	.	.	.	2	.	2	2
	1a	4a	4b	2a	5a	4c	5b	5c
2P	1a	2a	2a	1a	5a	2a	5b	5c
3P	1a	4a	4b	2a	5a	4c	5b	5c
5P	1a	4a	4b	2a	1a	4c	1a	1a
X.1	1	1	1	1	1	1	1	1
X.2	1	-1	-1	1	1	1	1	1
X.3	1	-1	1	1	1	-1	1	1
X.4	1	1	-1	1	1	-1	1	1
X.5	2	.	.	-2	2	.	2	2
X.6	8	.	.	.	3	.	-2	-2
X.7	8	.	.	.	-2	.	-2	3
X.8	8	.	.	.	-2	.	3	-2

□

Our conjecture is that, all irreducible characters of every Frobenius $\mathbb{Q}$-group vanish only on 2-elements. However the next example which we found in [1] shows that there are non-rational Frobenius groups which do not satisfy the condition, vanishing only on p-elements.

Example 2.8. For every prime $p \geq 5$, there exists a Frobenius group G with complement of order p and with abelian kernel such that G has some irreducible character which vanish on an element of order relatively prime to p.

We start by choosing a natural number m, coprime with p, such that p can be written as a linear combination of the prime factors of m, with non-negative integer coefficients. Note that this can be done as $p \geq 5$: take, for instance, $m = 6$.

Let A be the direct product of p-1 copies of a cyclic group C_m of order m and let $v_1, v_2, \cdots, v_{p-1}$ be generators of A. Let α be the automorphism of A defined by $v_i^\alpha = v_{i+1}$, for $1 \leq i < p-1$, and $v_{p-1}^\alpha = (v_1 \cdots v_{p-1})^{-1}$. Set $G = A :< \alpha >$. Then G is a Frobenius group with complement of order p and abelian kernel. However G has an irreducible character which vanishes on all elements of order p, as well as but also on the element v_1.

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PSEUDO-VALUATION MODULES

FOROOZAN KHOSHAYAND

ABSTRACT. The aim of this talk is to generalize the notion of pseudo-valuation to modules with prime submodule over arbitrary commutative rings. We generalize the notion of strongly prime ideal, as defined in [1], to the notion of strongly prime submodules.

1. INTRODUCTION

Throughout this talk, we consider commutative rings with identity and modules with prime submodule which are unitary. In [4], Hedstrom and Houston introduced pseudo-valuation domains. In [1], the study of pseudo-valuation domains was generalized to arbitrary rings.

In this talk, we generalize the notion of pseudo-valuation to modules over arbitrary commutative rings. In section 2, we define a pseudo-valuation module and study some properties of pseudo-valuation modules.

In section 3, we investigate some topological properties. Zariski spaces of modules were introduced in [6]. The prime spectrum of M, $Spec(M)$, is the set of all prime submodules of M. If N is a submodule of an $R-$module M, the variety of N, which is denoted by $V(N)$, is the set consisting of all prime submodules of M that contain N. An $R-$module M is called a module with Zariski topology (or Top-module), if the set of all varieties

$$\xi(M) := \{\ V(N)\ \mid\ N \subseteq M\}$$

is closed under finite unions, and hence constitutes the closed sets in a Zariski-like topology on $Spec(M)$. When $Spec(M) \neq \emptyset$, the map

$$\psi : Spec(M) \to Spec(R/Ann(M))$$

defined by $\psi(N) = (N : M)/Ann(M)$, for every $N \in Spec(M)$, will be called the natural map of $Spec(M)$. Also $\xi(M)$ is a $\xi(R)-$semimodule with respect to the following addition and scalar multiplication

$$V(N_1) + V(N_2) = V(N_1) \cap V(N_2) = V(N_1 + N_2)$$
$$V(I)V(N) = V(IN)$$

for all submodules N_1, N_2, N of M and all ideals I of R. Let k be a positive integer and let $N_i (1 \leq i \leq k)$ be submodules of M. Then $< V(N_1), \ldots, V(N_k) >= \{V(I_1)V(N_1) + \cdots + V(I_k)V(N_k)\ \mid\ I_1, \ldots, I_k \textit{ are ideals of } R\}$. A module M is

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called *Zariski-finite* if $\xi(M) =< V(N_1), \dots, V(N_k) >$, for some positive integer k and submodules $N_i (1 \leq i \leq k)$. In particular, if $k = 1$, M is called *Zariski-cyclic*.

2. Pseudo-valuation modules

Definition 2.1. Let M be an $R-$module. A prime submodule N of M is said to be *strongly prime* if Rm and rN are comparable (under inclusion) for all $r \in R$ and $m \in M$.

Proposition 2.2. *Let N be a strongly $P-$prime submodule of an $R-$module M. Then $N = PM$.*

Corollary 2.3. *Let M be an $R-$module. Then there is at most one strongly $P-$prime submodule of M, for every prime ideal P of R.*

Proposition 2.4. *Let M be a multiplication $R-$module and P be a strongly prime ideal of R. If $0 \neq PM$ then PM is a strongly prime submodule of M.*

Definition 2.5. Let M be an $R-$module with prime submodule. If every prime submodule of M is a strongly prime submodule, then M is called a *pseudo-valuation $R-$module.*

Proposition 2.6. *Let M be a pseudo-valuation $R-$module. Then $S^{-1}M$ is a pseudo-valuation $S^{-1}R-$module, for every multiplicatively closed subset S of R.*

Proposition 2.7. *Let R be a pseudo-valuation ring and M be a multiplication $R-$module. Then M is a pseudo-valuation $R-$module.*

Corollary 2.8. *Let R be a pseudo-valuation ring and I be a proper ideal of R. Then $\frac{R}{I}$ is a pseudo-valuation $R-$module.*

Proposition 2.9. *Let M be an $R-$module and N be a maximal submodule of M. If N is strongly prime, then M is a cyclic module.*

Corollary 2.10. *Every pseudo-valuation multiplication module is a cyclic module.*

Corollary 2.11. *Every finitely generated pseudo-valuation module is a cyclic module and so is a multiplication module.*

Theorem 2.12. *Let M be an $R-$module with a maximal submodule N. Then M is a pseudo-valuation module if and only if N is a strongly prime submodule of M.*

Proposition 2.13. *Let M be a pseudo-valuation $R-$module. If R satisfies the ascending (descending) chain condition on prime ideals then M satisfies the ascending (descending) chain condition on prime submodules.*

Corollary 2.14. *Let R be a Noetherian ring and M be a pseudo-valuation $R-$module. Then every prime submodules of M is finitely generated.*

Proposition 2.15. *Let M be a free module over integral domain R. If M has a basis with more than one element, then M is not a pseudo-valuation module.*

Corollary 2.16. *Let M be a free pseudo-valuation module over integral domain R. Then R is a pseudo-valuation ring.*

3. Properties of pseudo-valuation module

For a prime ideal P of R, the set of all $P-$prime submodules of an $R-$module M is denoted by $Spec_P(M)$. An $R-$module M is said to be *Zariski-bounded* if there exists a positive integer n such that for every prime ideal P of R, $\mid Spec_P(M) \mid \leq n$.

Proposition 3.1. *Every pseudo-valuation module is a Zariski-bounded module.*

Proposition 3.2. *Every pseudo-valuation module is a Top-module.*

Proposition 3.3. *Let M be a pseudo-valuation $R-$module. Then M has a injective natural map.*

We recall that a topological space X is said to be a T_0-space if for any pair of distinct points of X, there exists an open set which contains one of them but not the other. A topological space X is irreducible if the intersection of any two non-empty open sets of X is non-empty and X is said to be quasi-compact if every open cover of X has a finite subcover.

Proposition 3.4. *Let M be a pseudo-valuation $R-$module. Then $X = Spec(M)$ is an irreducible T_0-space.*

Proposition 3.5. *Let M be a pseudo-valuation module. Then*

$$\xi(M) = \{V(PM) \mid P \text{ is a prime ideal of } R\} \bigcup \{\emptyset, Spec(M)\}$$

Corollary 3.6. *Every pseudo-valuation module M is a Zariski-cyclic module.*

Corollary 3.7. *Let M be a pseudo-valuation module. Then $X = Spec(M)$ is a connected space.*

Corollary 3.8. *Let M be a pseudo-valuation $R-$module. Then $X = Spec(M)$ is a quasi-compact space.*

Let R be an integral domain with quotient field K. By [7], an $R-$module M is said to be integrally closed whenever $a^n m_n + \cdots + am_1 + m_0 = 0$ for some $n \in \mathbb{N}$ and $a \in K$ and $m_i \in M$, then $am_n \in M$.

Proposition 3.9. *Let N be a strongly prime submodule of an $R-$module M. Then N is an integrally closed $R-$module.*

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ON COMMUTATIVITY DEGREE OF FINITE GROUPS

H. KHOSRAVI

Abstract. In this talk we study commutativity degree, $d(G)$, the probability that a randomly chosen pair of elements of G commute, and show that for a finite group G, if p is the least prime divisor of $|G|$ then, $d(G) = \frac{p^2+p-1}{p^3}$ if and only if $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Also it is shown that for an arbitrary group G, $d(G) = \frac{p^2+p-1}{p^3}$ if and only if G is isoclinic to a group of order p^3. Finally we characterize, up to isoclinism, all groups G for which $\frac{11}{27} \leq d(G) < \frac{1}{2}$.

1. Introduction

Let G be a finite group. As it is pointed out by W. H. Gustafson [3], the probability that a randomly chosen pair of elements of G commute is

$$\frac{|\{(x,y) \in G \times G : xy = yx\}|}{|G|^2} = \frac{k(G)}{|G|},$$

where $k(G)$ is the number of conjugacy classes of G. We denote this probability by $d(G)$. It is perhaps surprising that there are no finite groups G with $\frac{5}{8} < d(G) < 1$, [3]. Ernest [1] showed that if H is a subgroup of G then $k(G) \leq |G : H|k(H)$. Therefore for an arbitrary subgroup H of G, $d(G) \leq d(H)$. Gallagher [2] proved that if N is a normal subgroup of a finite group G, then $k(G) \leq k(N)k(\frac{G}{N})$ and therefore $d(G) \leq d(N)d(\frac{G}{N})$. Lescot [4, 5] showed that if $d(G) > \frac{1}{2}$, then G is nilpotent and if $d(G) = \frac{1}{2}$ and G is not nilpotent, then $\frac{G}{Z(G)} \cong S_3$ and $G' \cong Z_3$.

2. Results

Lescot [6], upon by means of the concept of isoclinism, interpreted and even slightly improved all of the above results. In this paper he proved that if $G \sim H$, then $d(G) = d(H)$ and

Proposition 2.1. *Let G be any group; then there is a group G_1 isoclinic to G such that $Z(G_1) \subseteq G_1'$. If G is finite, so is any such G_1.*

Also he gave an upper bound for the commutativity degree of a finite $p-$groups. In fact he proved that $d(G) \leq \frac{p^2+p-1}{p^3}$ if G is a finite p-group. Moghaddam et. al[7], extended this result for an arbitrary finite group, where p is the least prime divisor of $|G|$. In this article, we prove that equality in the above result is hold if and only if

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$\dfrac{G}{Z(G)} \cong \mathrm{Z}_p \times \mathrm{Z}_p$. Also we characterize, up to isoclinism, all groups G with condition $d(G) = \frac{p^2+p-1}{p^3}$, where p is a prime number. In fact it is shown that

Theorem 2.2. *Let G be an arbitrary finite group. Then $d(G) = \dfrac{p^2+p-1}{p^3}$ if and only if G is isoclinic with a non abelian group of order p^3.*

Finally we prove that

Theorem 2.3. *Let G be a finite group. If $\frac{11}{27} \leq d(G) < \frac{1}{2}$, then one of the following holds*

(1) *G is isoclinic with a non-abeliab group of order 27, $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $d(G) = \frac{11}{27}$.*
(2) *$\frac{G}{Z(G)}$ is abelian, $|G'| = 4$ and $d(G) = \frac{1}{4}[1 + \frac{12}{|G|} + \frac{m}{|G|}]$, where $m = |\{x \in G||G : C_G(x)| = 2\}|$.*
(3) *G is isoclinic with a non-abelian group of order 16, $|G'| = 4$ and $d(G) = \frac{7}{16}$.*

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ON THE CATEGORY OF LOCAL HOMEOMORPHISMS WITH UNIQUE PATH LIFTING PROPERTY

MAJID KOWKABI[†]

ABSTRACT. In this talk, we discuss on the category of local homeomorphisms of topological spaces with unique path lifting property. We intend to find a classification of these local homeomorphisms similar to that of covering maps.

This is a joint work with Hamid Torabi[‡] and Behrooz Mashayekhy[††].

1. Introduction

Biss [2, Theorem 5.5] showed that for a connected, locally path connected space X, there is a 1-1 correspondence between its equivalent classes of connected covering spaces and the conjugacy classes of open subgroups of its fundamental group $\pi_1(X,x)$. There is a misstep in the proof of the above theorem. In fact, Biss assumed that every fibration with discrete fiber is a covering map which is not true in general.

Torabi et al.[7] pointed out the above misstep and gave the true classification of connected covering spaces of X according to open subgroups of the fundamental group $\pi_1(X,x)$. In fact, for a connected, locally path connected space X, there is a 1-1 correspondence between its equivalent classes of connected covering spaces and the conjugacy classes of subgroups of its fundamental group $\pi_1(X,x)$, with an open normal subgroup in $\pi_1^{qtop}(X,x)$. We know every covering map is a local homeomorphism. In this talk, we intend to study the category of local homeomorphisms $p:\tilde{X}\longrightarrow X$ for a fixed topological space $X$with unique path lifting property. Our main contribution is to find a classification for these local homeomorphisms.

2. Notations and preliminaries

For a topological space X, by a path in X we mean a continuous map $\alpha:[0,1]\to X$. The points $\alpha(0)$ and $\alpha(1)$ are called the initial point and the terminal point of α, respectively. A loop α is a path with $\alpha(0)=\alpha(1)$. For a path $\alpha:[0,1]\to X$, α^{-1} denotes a path such that $\alpha^{-1}(t)=\alpha(1-t)$, for all $t\in[0,1]$. Denote $[0,1]$ by I, two paths $\alpha,\beta:I\to X$ with the same initial and terminal points are called

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homotopic relative to end points if there exists a continuous map $F : I \times I \to X$ such that

$$F(t,s) = \begin{cases} \alpha(t) & s = 0 \\ \beta(t) & s = 1 \\ \alpha(0) = \beta(0) & t = 0 \\ \alpha(1) = \beta(1) & t = 1. \end{cases}$$

Homotopy relative to end points is an equivalent relation and the homotopy class containing a path α is denoted by $[\alpha]$. For paths $\alpha, \beta : I \to X$ with $\alpha(1) = \beta(0), \alpha * \beta$ denotes the concatenation of α and β which is a path from I to X such that

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t) & 0 \le t \le 1/2 \\ \beta(2t-1) & 1/2 \le t \le 1. \end{cases}$$

The set of all homotopy classes of loops relative to the end point x in X under the binary operation $[\alpha][\beta] = [\alpha * \beta]$ forms a group which is called the fundamental group of X and is denoted by $\pi_1(X, x)$ (see[6]). The set of all loops with initial point x in X is called the loop space of X denoted by $\Omega(X, x)$ (see [5]).

The quasitopological fundamental group $\pi_1^{qtop}(X, x)$ is the quotient space of the loop space $\Omega(X, x)$ equipped with the compact-open topology with respect to the function $\Omega(X, x) \to \pi_1(X, x)$ identifying path components (see [2]). It should be mentioned that$\pi_1^{qtop}(X, x)$ is a quasitopological group in the sense of [1] and it is not always a topological group (see [3],[4]).

Definition 2.1. [5] *Assume that X and $\tilde{X}$ are topological spaces. The continuous map $p : \tilde{X} \longrightarrow X$ is called a* **local homeomorphism** *if for every point $\tilde{x} \in \tilde{X}$ there exists an open set $\tilde{W}$ such that $\tilde{x} \in \tilde{W}$ and $p(\tilde{W}) \subset X$ is open and the restriction map $p|_{\tilde{W}} : \tilde{W} \longrightarrow p(\tilde{W})$ is a homeomorphism.*

Definition 2.2. *Let $p : \tilde{X} \longrightarrow X$ be a local homeomorphism and let $f : (Y, y_0) \to (X, x_0)$ be a continuous map with $f(y_0) = x_0$. Let $\tilde{x_0}$ be in the fiber over x_0. If there exist a continuous function $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x_0})$ such that $p \circ \tilde{f} = f$, then $\tilde{f}$ is called a lifting for f.*

Definition 2.3. *Assume that X and $\tilde{X}$ are topological spaces and $p : \tilde{X} \longrightarrow X$ is a continuous map. Let $\tilde{x_0}$ be in the fiber over x_0. The map p has "unique path lifting property" if for every path f in X, there exists a unique continuous function $\tilde{f} : (I, 0) \to (\tilde{X}, \tilde{x_0})$ with $p \circ \tilde{f} = f$.*

Let X be a fixed topological space. The set of all local homeomorphisms of X with unique path lifting property forms a category. In this category a morphism from $p : \tilde{X} \to X$ to $q : \tilde{Y} \to X$ is a continuous function $h : \tilde{X} \to \tilde{Y}$ such that $p = q \circ h$.

Definition 2.4. [6] *Let $\tilde{X}$ and X be topological spaces and let $p : \tilde{X} \longrightarrow X$ be continuous. An open set U in X is* **evenly covered** *by p if $p^{-1}(U)$ is a disjoint union of open sets S_i in $\tilde{X}$, called* **sheets**, *such that $p|_{S_i} : S_i \to U$ is a homeomorphism for every i.*

Definition 2.5. [6] *If X is a topological space, then an ordered pair $(\tilde{X}, p)$ is a* **covering space** *of X if:*

(1) *$\tilde{X}$ is a path connected topological space;*
(2) *$p : \tilde{X} \to X$ is continuous;*
(3) *each $x \in X$ has an open neighborhood $U = U_x$ that is evenly covered by p.*

3. Main Results

Theorem 3.1. **(Local Homeomorphism Homotopy Theorem for Paths)** *Let $(\tilde{X}, p)$ be a local homeomorphism of X with unique path lifting property. Consider the following diagram of continuous maps*

$$\begin{tikzcd} I \arrow[r, "\tilde{f}"] \arrow[d, "j"'] & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ I \times I \arrow[r, "F"] \arrow[ur, dashed, "\tilde{F}"] & (X, x_0) \end{tikzcd}$$

where $j(t) = (t, 0)$ for all $t \in I$. Then there exists a unique continuous map $\tilde{F} : I \times I \to \tilde{X}$ which makes the diagram commutative.

Theorem 3.2. **(Lifting Criterion)** *If Y is connected and locally path connected, $f : (Y, y_0) \to (X, x_0)$ is continuous and $p : \tilde{X} \to X$ is a local homeomorphism with unique path lifting property, where $\tilde{X}$ is path connected, then there exists a unique $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x_0})$ such that $p \circ \tilde{f} = f$ if and only if $f_*(\pi_1(Y, y_0) \subset p_*(\pi_1(\tilde{X}, \tilde{x_0}))$.*

Corollary 3.3. *If Y is simply connected, locally path connected and $p : \tilde{X} \to X$ is a local homeomorphism with unique path lifting property, where $\tilde{X}$ is path connected, then any continuous map $f : (Y, y_0) \to (X, x_0)$ has a lifting to $\tilde{X}$.*

Corollary 3.4. *Suppose X is connected, locally path connected and $p : \tilde{X} \to X$, $q : \tilde{Y} \to X$ are local homeomorphisms with unique path lifting property where $\tilde{X}$, $\tilde{Y}$ are path connected. If $p_*(\pi_1(\tilde{X}, \tilde{x_0})) = q_*(\pi_1(\tilde{Y}, \tilde{y_0}))$, then there exists a homeomorphism $h : (\tilde{Y}, \tilde{y_0}) \to (\tilde{X}, \tilde{x_0})$ such that $p \circ h = q$.*

Theorem 3.5. *Let $p : \tilde{X} \to X$ be a local homeomorphism with unique path lifting property and let $x_0, x_1 \in X$ and $f, g : I \to X$ be paths with $f(0) = g(0) = x_0$, $f(1) = g(1) = x_1$ and $\tilde{x_0} \in p^{-1}(x_0)$. If $F : f \simeq g$ rel $\dot{I}$ and $\tilde{f}, \tilde{g}$ are the lifting of f and g respectively with $\tilde{f}(0) = \tilde{x_0} = \tilde{g}(0)$, then $\tilde{F} : \tilde{f} \simeq \tilde{g}$ rel $\dot{I}$.*

Theorem 3.6. *Let $p : \tilde{X} \to X$ be a local homeomorphism with unique path lifting property where $\tilde{X}$ is path connected. If $x_0, x_1 \in X$, $Y_0 = p^{-1}(x_0)$ and $Y_1 = p^{-1}(x_1)$, then $|Y_0| = |Y_1|$.*

Theorem 3.7. *If X is connected, locally path connected and H is a subgroup of $\pi_1(X, x)$, then there exists a local homeomorphism $p : \tilde{X} \to X$ with unique path lifting property such that $p_*(\pi_1(\tilde{X}, \tilde{x})) = H$ if and only if H is an open subgroup of $\pi_1^{qtop}(X, x)$. Moreover there is a 1-1 correspondence between equivalent classes of local homeomorphisms of X (in category of local homeomorphism with unique path lifting property) and the conjugacy classes of open subgroups of the quasitopological fundamental group $\pi_1^{qtop}(X, x)$.*

Definition 3.8. *$p : \tilde{X} \to X$ is called a* **regular local homeomorphism** *if $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ is a normal subgroup of $\pi_1(X, x_0)$.*

Theorem 3.9. *Every regular local homeomorphism is a cover map.*

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SOME PROPERTIES OF NILPOTENT AND SOLVABLE LIE ALGEBRAS

SHOKUFEH LOTFI

Abstract. In this talk, we are going to explain some properties of Lie algebras and we prove that, if L is a locally solvable Lie algebra with a minimal ideal I, then I' is trivial. Also, we show that if L is a nilpotent Lie algebra and $I \neq 0$ is an ideal in L, then $I \cap Z(L) \neq 0$. Finally, it is shown that every Lie subalgebra of a finitely generated nilpotent Lie algebra L is finitely generated.

1. Introduction

Let F be a field. A *Lie algebra* over F is an F-vector space L, together with a bilinear map, the Lie bracket

$$L \times L \longrightarrow L, \quad (x, y) \longmapsto [x, y]$$

satisfying the following properties:

(i) $[x, x] = 0$ for all $x \in L$,
(ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$.

condition (ii) is known as the Jacobi identity and condition (i) implies $[x, y]{=}-[y, x]$ for all $x, y \in L$.

Throughout this talk, we use the term Lie algebra over some fixed field F and $[\,,]$ denotes the Lie bracket. We shall follow [1] and [3] for notation and terminology. If K be a vector subspace of L such that $[x, y] \in K$ for all $x, y \in K$, then K is a *subalgebra* of L and we write $K \leq L$. Also an *ideal* of a Lie algebra L is a subspace I of L such that $[x, y] \in I$ for all $x \in L$, $y \in I$ and we write $I \lhd L$.

The *derived series* of L is defined to be the series with terms

$$L^{(1)} = L' \text{ and } L^{(n)} = [L^{(n-1)}, L^{(n-1)}] \text{ for } n \geq 2.$$

Then $L \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \ldots$. As the product of ideals is an ideal, $L^{(n)}$ is an ideal of L.

Definition 1.1. *The Lie algebra L is said to be solvable if there exists an integer $m \geq 1$, such that $L^{(m)} = 0$.*

Let L be a solvable Lie algebra. If m is the least positive integer such that $L^{(m)} = 0$, then m is called *solvable length* of L. For example, the *Heisenberg* algebra L is solvable, where $L{=}\langle x, y, z{=}[x, y]\rangle$.

We define the *lower central series* of Lie algebra L to be the series with terms

$$L^1 = L \text{ and } L^n = [L^{n-1}, L] \text{ for } n \geq 2.$$

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Then $L = L^1 \supseteq L^2 \supseteq \ldots$. As the product of ideals is an ideal, L^n is an ideal of L.

Definition 1.2. *The Lie algebra L is said to be nilpotent if there exists an integer $m \geq 1$, such that $L^m = 0$.*

Let L be a Lie algebra. Then the *upper central series* of L are defined as follows:

$$0 = Z_0(L) \subseteq Z_1(L) = Z(L) \subseteq \cdots \subseteq Z_n(L) \subseteq \cdots,$$

where $Z_n(L)/Z_{n-1}(L) = Z\big(L/Z_{n-1}(L)\big)$, for $n \geq 1$.

The Lie algebra L is said to be *nilpotent of class k*, if $Z_k(L) = L$ and $Z_{k-1}(L) \neq L$ or $L^{k+1} = 0$ and $L^k \neq 0$.

2. Main Results

In this section we shall investigate the properties of nilpotent and solvable Lie algebras, with particular regard to analogus conditions in group theory (see [2]).

The following result generalizes a theorem of Malcev and Mclain [6].

Theorem 2.1. *If L is a locally solvable Lie algebra with minimal ideal I, then I' is trivial.*

Proof. Suppose contrary that n_1 and n_2 are elements of I such that $n = [n_1, n_2] \neq 0$. Since $n \in I$, by minimality of I, we must have, I is ideal generated by n in L. So that there are elements $l_1, \ldots, l_m$ of I such that $n_1, n_2 \in \langle [n, l_1], \ldots, [n, l_m] \rangle$. Let H be the ideal generated by $\{n_1, n_2, l_1, \ldots, l_m\}$, a solvable Lie algebra, and consider $A = \langle n, [n, n_1], [n, n_2], [n, l_1], \ldots, [n, l_n] \rangle$ in H. Since A contains n_1 and n_2, the element n belongs to A'. Consequently, $A \leq A'$. However, this means that $A = 0$, because H is solvable. □

The following result generalizes a theorem of Hall [4].

Theorem 2.2. *If L is a nilpotent Lie algebra and $I \neq 0$ is an ideal in L, then $I \cap Z(L) \neq 0$.*

Proof. Since L is nilpotent, so there exists an integer $m \geq 1$ such that $Z_m(L) = L$. Suppose that i is the least positive integer such that $I \cap Z_i(L) \neq 0$. Now,

$$[I \cap Z_i(L), L] \subseteq [Z_i(L), L] \subseteq Z_{i-1}(L). \qquad (*)$$

Since I and $Z_i(L)$ are ideals in L, so $I \cap Z_i(L)$ is an ideal in L, thus

$$[I \cap Z_i(L), L] \subseteq I \cap Z_i(L) \subseteq I. \qquad (**)$$

Hence by $(*)$ and $(**)$, we have $[I \cap Z_i(L), L] \subseteq I \cap Z_{i-1}(L) = 0$ and $0 \neq I \cap Z_i(L) \leq I \cap Z(L)$. □

Theorem 2.3. *If L is a locally nilpotent Lie algebra with minimal ideal I, then $I \leq Z(L)$.*

Proof. Suppose that this is false and $I \nleq Z(L)$. If $I \nleq Z(L)$, there exist a in I and l in L such that $b = [a, l] \neq 0$. Since $b \in I$, we have $I = \langle b \rangle$ by minimality of I. Hence $a \in \langle [b, l_1], \ldots, [b, l_n] \rangle$ for certain $l_i \in L$. Let H be the ideal generated by $\{a, l, l_1, \ldots, l_n\}$, a nilpotent Lie algebra, and set A to be the ideal generated by $\langle a, [a, l], [a, l_1], \ldots, [a, l_n] \rangle$ in H. Then $b \in [A, H]$, so that $[b, l_i] \in [A, H]$ and consequently, $a \in [A, H]$. Hence $A = [A, H]$ and $A = [[A, H], H] = [\ldots [A, H], \ldots, H]$. Since H is nilpotent, there exists an integer $r \geq 1$ such that $L^r = 0$, therefore $[A, {}_rH] = 0$, it follows that $A = 0$ and $a = 0$. But this means that $b = 0$. $\square$

The following lemma is needed for the proof of the next theorem.

Lemma 2.4. *Let $\{x_\lambda \,|\, \lambda \in \Lambda\}$ be a set of generators for the Lie algebra L. Then for each $i \geq 1$, L^i/L^{i+1} is generated by elements of the form $[b_1, \ldots, b_i] + L^{i+1}$, where $b_j \in \{x_\lambda \,|\, \lambda \in \Lambda\}$ for $1 \leq j \leq i$.*

Finally we consider a sufficient condition for a finitely generated Lie algebra to be finite-dimensional, cf. [5].

Theorem 2.5. *If L is a finitely generated nilpotent Lie algebra, then any Lie subalgebra of L is finitely generated.*

Proof. Let I be a subalgebra of L. Since L is nilpotent, there exists an integer $m \geq 1$, such that $L^m = 0$. Put $I_k = L^k \cap I$ for each $1 \leq i \leq m$. So for each $1 \leq i \leq m-1$, we have

$$\frac{I_i}{I_{i+1}} = \frac{L^i \cap I}{L^{i+1} \cap I} = \frac{L^i \cap I}{L^{i+1} \cap \big(L^i \cap I\big)} \cong \frac{L^{i+1} + \big(L^i \cap I\big)}{L^{i+1}} \subseteq \frac{L^i}{L^{i+1}}$$

since L^i/L^{i+1} is finitely generated abelian, hence I_i/I_{i+1} is finitely generated. Since $I_m = 0$, hence I_{m-1} and I_{m-2}/I_{m-1} are both finitely generated, therefore I_{m-2} is finitely generated. By continuing this argument turns out $I_1 = I$ is finitely generated, as required. $\square$

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ON RELATIONS BETWEEN LOCAL COHOMOLOGY MODULES

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Abstract. Let R be a Noetherian ring, I, J two ideals of R, M an R-module and t an integer. Let S be a Serre subcategory of the category of R-modules satisfying the condition C_I and $\mathfrak{a} \in \tilde{\mathrm{W}}(I,J)$. It is shown that if $\mathrm{Ext}^j_R(R/\mathfrak{a}, H^i_{I,J}(M)) \in S$ for all $i < t$ and all $j < t - i$, then $H^i_{\mathfrak{a}}(M) \in S$ for all $i < t$. Let S be the class of all R-modules N with $\dim_R N \leq k$, where k is an integer. It is proved that if $H^i_{\mathfrak{a}}(M) \in S$ for all $i < t$ and all $\mathfrak{a} \in \tilde{\mathrm{W}}(I,J)$, then $H^i_{I,J}(M) \in S$ for all $i < t$. It follows that $\inf\{i : H^i_{I,J}(M) \notin S\} = \inf\{\inf\{i : H^i_{\mathfrak{a}}(M) \notin S\} : \mathfrak{a} \in \tilde{\mathrm{W}}(I,J)\}$.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring with non-zero identity, I and J are two ideals of R, M is an R-module and t is an integer. For notations and terminologies not given in this paper, the reader is referred to [2], [3] and [5] if necessary.

The local cohomology theory has been an significant tool in commutative Algebra and Algebraic Geometry. The local cohomology modules with respect to a system of ideals was introduced by Bijan-Zadeh, in [1]. As a special case of these extended modules, Takahashi, Yoshino and Yoshizawa defined the local cohomology modules with respect to a pair of ideals; see [5]. The set of elements x of M such that $I^t x \subseteq Jx$, for some positive integer t, is said to be (I,J)-torsion submodule of M and is denoted by $\Gamma_{I,J}(M)$. Let $\tilde{\mathrm{W}}(I,J) = \{\mathfrak{a} \leq R : I^t \subseteq J + \mathfrak{a}$ for some positive integer $t\}$. Then $x \in \Gamma_{I,J}(M)$ if and only if $\mathrm{Ann}_R(x) \in \tilde{\mathrm{W}}(I,J)$. It is easy to see that $\Gamma_{I,J}$ is a covariant, R-linear functor from the category of R-modules to itself. For an integer i, the local cohomology functor $H^i_{I,J}$ with respect to (I,J), is defined to be the i-th right derived functor of $\Gamma_{I,J}$. Also $H^i_{I,J}(M)$ is called the i-th local cohomology module of M with respect to (I,J). If $J = 0$, then $H^i_{I,J}$ coincides with the ordinary local cohomology functor H^i_I.

One of the important problems is to determine relations between ordinary local cohomology modules and local cohomology modules with respect to a pair of ideals. In this direction, the authors in [5, Theorem 3.2] proved that

$$H^i_{I,J}(M) \cong \varinjlim_{\mathfrak{a} \in \tilde{W}(I,J)} H^i_{\mathfrak{a}}(M)$$

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for any integer i. Let S be a Serre subcategory of the category of R-modules satisfying the condition C_I and M be a finitely generated R-module. In [4, Theorem 2.11] it is showen that if $H^i_{I,J}(M) \in S$ for all $i < t$, then $H^i_I(M) \in S$ for all $i < t$. We improve this result and show that if $\mathrm{Ext}^j_R(R/\mathfrak{a}, H^i_{I,J}(M)) \in S$ for all $i < t$ and all $j < t-i$, then $H^i_{\mathfrak{a}}(M) \in S$ for all $i < t$, where M is an arbitrary R-module and $\mathfrak{a} \in \tilde{\mathrm{W}}(I,J)$.

Let S be the class of all R-modules N with $\dim_R N \leq k$, where k is an integer. We show that if $H^i_{\mathfrak{a}}(M) \in S$ for all $i < t$ and all $\mathfrak{a} \in \tilde{\mathrm{W}}(I,J)$, then $H^i_{I,J}(M) \in S$ for all $i < t$. As a consequence, it follows that

$$\inf\{i : H^i_{I,J}(M) \notin S\} = \inf\{\inf\{i : H^i_{\mathfrak{a}}(M) \notin S\} : \mathfrak{a} \in \tilde{\mathrm{W}}(I,J)\}.$$

2. Main Results

Definition 2.1. *A full subcategory of the category of R-modules is said to be a Serre subcategory, if it is closed under taking submodules, quotients and extensions. A Serre subcategory S is said to satisfy the condition C_I, if for any I-torsion R-module M, $(0 :_M I) \in S$ implies that $M \in S$.*

The classes of zero modules, Artinian modules, I-cofinite Artinian modules, modules with finite support and the class of R-modules N with $\dim_R N \leq k$, where k is a non-negative integer, are Serre subcategories of the category of R-modules satisfying the condition C_I. In the rest of the paper, S denotes a Serre subcategory of the category of R-modules.

Theorem 2.2. *Let $\mathfrak{a} \in \tilde{\mathrm{W}}(I,J)$. If $Ext^{t-i}_R(R/\mathfrak{a}, H^i_{I,J}(M)) \in S$ for all $i \leq t$, then $Ext^t_R(R/\mathfrak{a}, M) \in S$.*

Corollary 2.3. *Let $\mathfrak{a} \in \tilde{\mathrm{W}}(I,J)$. If $Ext^j_R(R/\mathfrak{a}, H^i_{I,J}(M)) \in S$ for all $i < t$ and all $j < t-i$, then $Ext^i_R(R/\mathfrak{a}, M) \in S$ for all $i < t$.*

The following result is ageneralization of [4, Theorem 2.11].

Corollary 2.4. *Let S satisfy the condition C_I and $\mathfrak{a} \in \tilde{\mathrm{W}}(I,J)$. If $Ext^j_R(R/\mathfrak{a}, H^i_{I,J}(M)) \in S$ for all $i < t$ and all $j < t-i$, then $H^i_{\mathfrak{a}}(M) \in S$ for all $i < t$.*

Corollary 2.5. *If S satisfies the condition C_I, then*

$$\inf\{i : H^i_{I,J}(M) \notin S\} \leq \inf\{\inf\{i : H^i_{\mathfrak{a}}(M) \notin S\} : \mathfrak{a} \in \tilde{\mathrm{W}}(I,J)\}.$$

We can replace the above inequality by an equality for a special Serre class. In this direction, we need to the following theorem.

Theorem 2.6. *Let S be the class of all R-modules N with $\dim_R N \leq k$, where k is an integer. Let $Ext^{t+1-i}_R(R/\mathfrak{a}, H^i_{I,J}(M)) \in S$ for all $i < t$ and all $\mathfrak{a} \in \tilde{\mathrm{W}}(I,J)$, and $Ext^t_R(R/\mathfrak{a}, M) \in S$ for all $\mathfrak{a} \in \tilde{\mathrm{W}}(I,J)$. Then $H^t_{I,J}(M) \in S$.*

Corollary 2.7. *Let S be the class of all R-modules N with $\dim_R N \leq k$, where k is an integer. If $Ext^i_R(R/\mathfrak{a}, M) \in S$ for all $i < t$ and all $\mathfrak{a} \in \tilde{\mathrm{W}}(I,J)$, then $H^i_{I,J}(M) \in S$ for all $i < t$.*

Corollary 2.8. *Let S be the class of all R-modules N with $\dim_R N \leq k$, where k is an integer. If $H^i_{\mathfrak{a}}(M) \in S$ for all $i < t$ and all $\mathfrak{a} \in \tilde{\mathrm{W}}(I,J)$, then $H^i_{I,J}(M) \in S$ for all $i < t$.*

Corollary 2.9. *Let S be the class of all R-modules N with $\dim_R N \leq k$, where k is an integer. Then the following statements are equivalent:*

(i) $H^i_{I,J}(M) \in S$ *for all* $i < t$;

(ii) $H^i_{\mathfrak{a}}(M) \in S$ *for all* $i < t$ *and all* $\mathfrak{a} \in \tilde{\mathrm{W}}(I, J)$.

Corollary 2.10. *Let S be the class of all R-modules N with $\dim_R N \leq k$, where k is an integer. Then*

$$\inf\{i : H^i_{I,J}(M) \notin S\} = \inf\{\inf\{i : H^i_{\mathfrak{a}}(M) \notin S\} : \mathfrak{a} \in \tilde{\mathrm{W}}(I, J)\}.$$

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ON THE CATEGORY OF β-TOPOLOGICAL HYPERGROUPS

H. MOMENAEE KERMANI†

Abstract. β-topological hypergroups were recently defined by the authors in the paper (*β-topological hypergroups*, to appear in Annals of the Alexandru Ioan Cuza University) as a generalization of topological groups. In this talk, we study β-topological hypergroups from categorical point of view. We show that there exists a full subcategory of the category of β-topological hypergroups which is topological over the category of hypergroups.

This is a joint work with S. Sh. Mousavi‡.

1. Introduction

In recent decades hypergroups have been used in several branches of mathematics such as geometry, graph theory, probability, etc. As a generalization of groups, it is reasonable to study hypergroups from geometric point of view. There are several important hypergroups, such as the hypergroup of joining points in the plane, whose hyperoperations are somehow "continuous". This fact makes the investigation on hypergroups with a topological structure more essential. In [3] the authors introduced the notion of β-topological hypergroups and here we show that there exists a full subcategory of the category of β-topological hypergroups which is topological over the category of hypergroups.

Let H be a nonempty set and $\mathscr{P}^*(H)$ be the set of all nonempty subsets of H and let $\circ$ be a hyperoperation or join operation on H, that is, $\circ$ is a function from $H \times H$ into $\mathscr{P}^*(H)$. The join operation is extended to subsets of H in a natural way, that is $A \circ B = \bigcup\{a \circ b \mid a \in A, b \in B\}$. The notation aA is used for $\{a\} \circ A$ and Aa for $A \circ \{a\}$. Generally, the singleton $\{a\}$ is identified with its member a. The structure $(H, \circ)$ is called a *hypergroup* if $aH = Ha = H$ for all $a \in H$ and $a(bc) = (ab)c$ for all $a, b, c \in H$. Suppose $(H, \circ)$ and $(H', \circ')$ are two hypergroups. A function $f : H \longrightarrow H'$ is called a *homomorphism* if $f(a \circ b) \subseteq f(a) \circ' f(b)$ for all a and b in S. Let R be an equivalence relation on H. For all pairs (A, B) of non-empty subsets of H, we set

$$A \overset{=}{R} B \Leftrightarrow a \, R \, b, \quad \forall a \in A, \forall b \in B.$$

The relation R is called *strongly regular on the left* (*on the right*) if $x \, R \, y \Rightarrow a \circ x \overset{=}{R} a \circ y$ ($x \, R \, y \Rightarrow x \circ a \overset{=}{R} y \circ a$ respectively), for all $(x, y, a) \in H^3$. Moreover, R is called *strongly regular* if it is strongly regular on the right and on the left.

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If R is a strongly regular relation, it is well known that the quotient $\frac{H}{R}$ is a group under the operation

$$R(x) \otimes R(y) = R(z), \quad \forall z \in x \circ y.$$

For all $n > 1$ define the relation β_n on H as follows:

$$a\, \beta_n\, b \Leftrightarrow \exists (x_1, ..., x_n) \in H^n \ : \ \{a, b\} \subseteq \prod_{i=1}^{n} x_i$$

and $\beta = \bigcup_{i=1}^{n} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation. It is known [2] that β is a strongly regular relation. In fact, β is the least equivalence relation on H such that the quotient $\frac{H}{\beta}$ is a group. This group is called the *fundamental group* of H. The equivalence class containing x is denoted by $\frac{x}{\beta}$ and the identity element of the group $\frac{H}{\beta}$ is denoted by $\frac{e}{\beta}$.

The nonempty subset A of H is called a *complete part* of H if for all $n \geqslant 2$ and for all $(x_1, x_2, ..., x_n) \in H^n$ the following implication holds:

$$\prod_{i=1}^{n} x_i \bigcap A \neq \emptyset \Rightarrow \prod_{i=1}^{n} x_i \subseteq A.$$

Let $\varphi_{_H} : H \longrightarrow \frac{H}{\beta}$ be the canonical projection. The kernel of $\varphi_{_H}$ is called the *heart* (or core) of H and is denoted by $\omega_{_H}$. Note that $\omega_{_H}$ is a complete part.

2. β-topological hypergroup

In this section, we introduce the notion of β-topological hypergroupgroup.

Definition 2.1. (see [3]) *Suppose that $(H, \circ)$ is a hypergroup and A is a nonempty subset of H. β-inverse of A denoted by A_β^{-1} is defined by:*

$$A_\beta^{-1} :\overset{def}{=} \{h \in H \mid h \circ a \subseteq \omega_{_H} \text{ for some } a \in A\}.$$

For $a \in A$, $\{a_\beta^{-1}\}$ is denoted by a_β^{-1}.

It is not hard to see that if $h \circ a \subseteq \omega_{_H}$, then $a \circ h \subseteq \omega_{_H}$.

Proposition 2.2. (see [3]) *Suppose that $(H, \circ)$ is a hypergroup and A, B are nonempty subsets of H. Then A_β^{-1} is a nonempty complete part of H.*

Definition 2.3. (see [3]) *The hypergroup $(H, \circ)$ together with the topology $\tau_{_H}$ is called a β-topological hypergroup, if the following conditions hold.*

($\mathcal{TH} - 1$) If $x \circ y \subseteq U$ for $U \in \tau_{_H}$ and $x, y \in H$, then there exist open sets V_x and V_y containing x and y respectively such that $V_x \circ V_y \subseteq U$;

($\mathcal{TH} - 2$) For each $U \in \tau_{_H}$ if $\omega_{_H} \subseteq U$, then there exists $V \in \tau_{_H}$ such that V is a complete part and $V \circ V_\beta^{-1} \subseteq U$.

Theorem 2.4. (see [3]) *Let $(H, \circ, \tau_{_H})$ be a β-topological hypergroup. Then $(\frac{H}{\beta}, \otimes, \tau_{_H}^{\beta})$ is a topological group.*

Example 2.5. *Suppose that* $(G, \cdot, \tau_G)$ *is a topological group and* $(H, \circ)$ *is a hypergroup. For all* $g \in G$*, put* $H_g = H \times \{g\}$ *and define the hypergroup* $(K, \circ)$ *where* $K = \bigcup_{g\in G} H_g$ *and* $k_1 \circ k_2 = \{(h, g_1 \cdot g_2) \mid h \in h_1 \circ h_2\}$ *for all* $k_1 = (h_1, g_1)$ *and* $k_2 = (h_2, g_2)$ *in* K*. Define the topology* τ_K *on* K *as follows:*

$$\mathcal{U} \in \tau_K \text{ if and only if } \mathcal{U} = \bigcup_{g\in U} H_g \text{ for some } U \in \tau_G.$$

Therefore, $(K, \circ, \tau_K)$ *is a* β*-topological hypergroup and* φ_K *is an open map.*

As a special case of example 2.5 we introduce a β-topological hypergroup as follows.

Example 2.6. *Let a topological group* $(G, \cdot, \tau)$ *together with a normal closed subgroup* H *be given. Define the set* $\mathcal{X}$ *as follows:*

$$\mathcal{X} :\overset{def}{=} \{U \cdot H \mid U \in \tau\}.$$

It is easy to see that $\bigcup \mathcal{X} = G$ *and for all* $U_1 \cdot H$ *and* $U_2 \cdot H$ *in* $\mathcal{X}$ *we have* $(U_1 \bigcap U_2) \cdot H \subseteq U_1 \cdot H \bigcap U_2 \cdot H$*. Let* τ_G *be the topology on* G *generated by* $\mathcal{X}$*. The mapping* $* : G \times G \longrightarrow \mathscr{P}^*(H)$ *defined by* $x \circ y :\overset{def}{=} xyH$ *is a hyperoperation on* G *and* $(G, \circ)$ *is a hypergroup. In this case* $\omega_G = H$ *and hence* $\frac{G}{\beta} = \frac{G}{H}$*. The topology* τ_G^β *on* $\frac{G}{\beta}$ *is exactly the quotient topology on* $\frac{G}{H}$ *induced by canonical projection* $\pi_G : G \longrightarrow \frac{G}{H}$ *. Note that for all* $x \in G$*,* $\varphi_G(x) = \pi_G(x)$*. We can easily see that* $(G, \circ, \tau_G)$ *is a* β*-topological hypergroup.*

3. Category of β-topological hypergroup

We know that the category of topological groups is topological over the category of groups. So the main objective of this section is to obtain a full subcategory of β-topological hypergroups such that it is topological over the category of hypergroups.

Suppose that $\mathscr{C} \xrightarrow{\mathbb{U}} \mathscr{D}$ is a faithful functor. let a $\mathbb{U}$-structured source

$$\{\, Y \xrightarrow{f_i} \mathbb{U}(X_i\,)\}$$

in $\mathscr{D}$ which is not necessarily assumed to be small, be given. Let $\frac{\{f_i\}}{\mathscr{D}}$ denote the category whose objects are objects $Z \in \mathscr{C}$ equipped with a morphism $h : \mathbb{U}(Z) \longrightarrow Y$ in $\mathscr{D}$, and a source $\{\, Z \xrightarrow{g_i} X_i \,\}$ in $\mathscr{C}$, such that $\mathbb{U}(g_i) = f_i \circ h$ for all i. (Its morphisms are, of course, morphisms in $\mathscr{C}$ commuting with the structure.) A semi-initial lift of $\{\, Y \xrightarrow{f_i} \mathbb{U}(X_i\,)\}$ is a terminal object of $\frac{\{f_i\}}{\mathscr{D}}$. If the corresponding h is an isomorphism, then it is called an initial lift.

Definition 3.1. [1] *A functor* $\mathscr{C} \xrightarrow{\mathbb{U}} \mathscr{D}$ *is called topological or* $(\mathscr{C}, \mathbb{U})$ *is topological over* $\mathscr{D}$ *provided that every* $\mathbb{U}$*-structured source* $\{\, Y \xrightarrow{f_i} \mathbb{U}(X_i\,)\}$ *has an initial lift.*

Suppose that $(H, \circ, \tau_{_H})$ and $(K, \circ', \tau_{_K})$ are two β-topological hypergroups. A morphism

$$(H, \circ, \tau_{_H}) \xrightarrow{f} (K, \circ', \tau_{_K})$$

is a homomorphism $(H, \circ) \xrightarrow{f} (K, \circ')$ such that $(H, \tau_{_H}) \xrightarrow{f} (K, \tau_{_K})$ is a continuous map.

The collection of β-topological hypergroups together with morphisms forms a category, which is denoted by $TopHyprg$. The full subcategory of $TopHyprg$ whose objects are β-topological hypergroups such that open sets are complete parts is denoted by $CTopHyprg$.

It is easy to see that $\mathcal{U} : CTopHyprg \longrightarrow Hyprg$ defined by

$$(H, \circ, \tau_{_H}) \xrightarrow{f} (K, \circ', \tau_{_K}) \xmapsto{\mathcal{U}} (H, \circ) \xrightarrow{f} (K, \circ')$$

is a forgetful functor.

Theorem 3.2. *$(CTopHyprg, \mathcal{U})$ is topological over the category $Hyprg$.*

Proof. Let $\{\ (H, \circ) \xrightarrow{f_i} \mathcal{U}(H_i, \circ_i, \tau_{_{H_i}}) = (H_i, \circ_i)\ \}_{i \in I}$ be given. Let

$$\Gamma \overset{\text{def}}{:=} \{\tau \subseteq P(H) \mid (H, \tau) \text{ be a topological space such that } f_i \text{ is continuous for every } i\}$$

and put $\tau_{_H} \overset{\text{def}}{:=} \bigcap\limits_{\tau \in \Gamma} \tau$. So $(H, \tau_{_H})$ is a topological space such that $\tau_{_H} \in \Gamma$ and $\{f_i^{-1}(U_i) \mid U_i \in \tau_{_{H_i}}\}_{i \in I}$ is a sub-bases for $\tau_{_H}$. Thus every open set in $\tau_{_H}$ is a complete part. Let $x, y \in H$ and $U \in \tau_{_H}$ such that $x \circ y \in U$ be given. Therefore there exist $U_{i_j} \in \tau_{_{H_{i_j}}}$ for $1 \leq j \leq n$ such that $\bigcap\limits_{j=1}^{n} f_{i_j}^{-1}(U_{i_j}) \subseteq U$ and $x \circ y \bigcap [\bigcap\limits_{j=1}^{n} f_{i_j}^{-1}(U_{i_j})] \neq \emptyset$. So $x \circ y \subseteq \bigcap\limits_{j=1}^{n} f_{i_j}^{-1}(U_{i_j})$ and hence for all $1 \leq j \leq n$, there exist V_{i_j} and V'_{i_j} containing x and y respectively such that $V_{i_j} \circ_{i_j} V'_{i_j} \subseteq f_{i_j}^{-1}(U_{i_j})$. Put $V \overset{\text{def}}{:=} \bigcap\limits_{j=1}^{n} V_{i_j}$ and $V' \overset{\text{def}}{:=} \bigcap\limits_{j=1}^{n} V'_{i_j}$. Thus V and V' belong to $\tau_{_H}$ and contain x and y respectively such that $V \circ V' \subseteq \bigcap\limits_{j=1}^{n} f_{i_j}^{-1}(U_{i_j}) \subseteq U$.

Now suppose that $U \in \tau_{_H}$ such that $\omega_{_H} \subseteq U$ is given. As above there exist $U_{i_j} \in \tau_{_{H_{i_j}}}$ for $1 \leq j \leq n$ such that $\bigcap\limits_{j=1}^{n} f_{i_j}^{-1}(U_{i_j}) \subseteq U$ and $\omega_{_H} \bigcap [\bigcap\limits_{j=1}^{n} f_{i_j}^{-1}(U_{i_j})] \neq \emptyset$. Therefore for each $1 \leq j \leq n$, $\omega_{_{H_{i_j}}} = f_{i_j}(\omega_{_H}) \cap U_{i_j} \neq \emptyset$ and hence $\frac{e_{i_j}}{\beta} \in \varphi_{_{H_{i_j}}}$. Thus for each $1 \leq j \leq n$ there exists $\mathcal{V}_{i_j}$ containing $\frac{e_{i_j}}{\beta}$ such that $\mathcal{V}_{i_j} \otimes_{i_j} \mathcal{V}_{i_j}{}^{-1} \subseteq \varphi_{_{H_{i_j}}}(U_{i_j})$. For each $1 \leq j \leq n$ define $V_{i_j} \overset{\text{def}}{:=} \varphi_{_{H_{i_j}}}^{-1}(\mathcal{V}_{i_j})$. So for each $1 \leq j \leq n$, $\omega_{_{H_{i_j}}} \subseteq V_{i_j} \in \tau_{_{H_{i_j}}}$ and $V_{i_j} \circ_{i_j} (V_{i_j})_\beta^{-1} \subseteq U_{i_j}$. Put $V \overset{\text{def}}{:=} \bigcap\limits_{j=1}^{n} f_{i_j}^{-1}(V_{i_j})$ and since $\omega_{_H} \subseteq f_{i_j}^{-1}(U_{i_j})$ for each $1 \leq j \leq n$, we have $\omega_{_H} \subseteq V$ and $V \circ V_\beta^{-1} \subseteq \bigcap\limits_{j=1}^{n} f_{i_j}^{-1}(V_{i_j}) \subseteq U$. Thus $(H, \circ, \tau_{_H}) \in CTopHyprg$. It is easy to see that the family $\{\ (H, \circ, \tau_{_H}) \xrightarrow{g_i} (H_i, \circ_i, \tau_{_{H_i}})\ \}_{i \in I}$ such that for each $i \in I$, g_i is defined by

$g_i(x) = f_i(x)$ is in $CTopHyprg$. Put $h \stackrel{\text{def}}{:=} id_{(H,\circ)} : \mathcal{U}(H, \circ, \tau_H) = (H, \circ) \longrightarrow (H, \circ)$. So $\mathcal{U}(g_i) = f_i \circ h$ and h is a terminal object of $\frac{\{f_i\}}{Hyprg}$.

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WHEN COSINGULAR MODULES ARE DISCRETE

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Abstract. In this talk we introduce the concepts of CD-rings and CD-modules. Let R be a ring and M an R-module. We call R (M), a CD-ring (CD-module) in the case where every (M-)cosingular R-module (in $\sigma[M]$) is discrete. Let R be a ring such that the class of cosingular R-modules is closed under factor modules. It is proved that R is a CD-ring if and only if every cosigular R-module is semisimple. The relations of CD-rings are investigated with GV-rings, SC-rings and CP-rings. Let R be a semilocal ring. It is shown that R is right CD if and only if R is left SC with $Soc(_RR)$ essential in $_RR$.

This is a joint work with Samira Asgari[††] and Yahya Talebi[‡].

1. Introduction

Throughout this paper, R is always an associative ring with unit and all modules are unitary right R-modules, unless otherwise stated. Let M be an R-module. An R-module N is generated by M or M-generated if there exists an epimorphism $f : M^{(A)} \to N$ for some index set A. An R-module N is said to be *subgenerated by* M if N is isomorphic to a submodule of an M-generated module. We denote by $\sigma[M]$ the full subcategory of the right R-modules whose objects are all right R-modules subgenerated by M (see [11]). A submodule L of M is *essential in* M denoted by $L \leq_e M$, if for every nonzero submodule K of M, $L \cap K \neq 0$. As a dual concept, a submodule N of a module M is called *small* in M (denoted by $N \ll M$) if for every proper submodule L of M, $N + L \neq M$. A module M is called *hollow* if every proper submodule of M is small in M.
$Rad(M)$, $Soc(M)$, $Z(M) = \{x \in M \mid xI = 0, I \leq_e R_R\}$ and $E(M)$ denote the Jacobson radical, the socle, the singular submodule and the injective envelope of M, respectively. Let M be a module. The notations $N \leq M$ and $N \leq_{\oplus} M$ will denote a submodule and a direct summand of M, respectively.

A module N is said to be small if there exists a module L such that $N \ll L$. It is well-known that a module is small if and only if it is small in its injective envelope. We say that a submodule N of a module M *lies above* a direct summand K of M, if $N/K \ll M/K$. Let N and L be submodules of M. N is called a supplement of L in

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M if it is minimal with respect to the property $M = N + L$, equivalently, $M = N + L$ and $N \cap L \ll N$. M is called supplemented if for each submodule A of M, there exists a submodule B of M such that $M = A + B$ and $A \cap B \ll B$. Any module M is called *amply supplemented* if for any two submodules A and B with $M = A + B$, A contains a supplement of B in M. Recall that M is called H-supplemented provided for every submodule N of M, there exists a direct summand D of M such that $\frac{N+D}{N} \ll \frac{M}{N}$ and $\frac{N+D}{D} \ll \frac{M}{D}$. Also M is called $\oplus$-supplemented in the case where for every $N \leq M$, there exists a direct summand K of M such that $M = N + K$ and $N \cap K \ll K$.

In [8], Talebi and Vanaja defined $\overline{Z}_M(N)$ as a dual of M-singular submodule as follows: $\overline{Z}_M(N) = Rej(N, \mathcal{MS}) = \bigcap\{Kerf \mid f : N \to S, S \in \mathcal{MS}\} = \bigcap\{U \subseteq N \mid N/U \in \mathcal{MS}\}$ where $\mathcal{MS}$ denotes the class of all M-small modules. They called N an *M-cosingular* (*non-M-cosingular*) module if $\overline{Z}_M(N) = 0$ ($\overline{Z}_M(N) = N$). Clearly every M-small module is M-cosingular. We should note that cosingular and non-cosingular concepts mean R-cosingular and non-R-cosingular.

Keskin and Tribak in [4], introduced and studied modules M such that every M-cosingular module in $\sigma[M]$ is projective in $\sigma[M]$. They called such modules $COSP$. They investigated some general properties of $COSP$-modules. $COSP$-modules were also characterized when every injective module in $\sigma[M]$ is amply supplemented. Finally they showed that a $COSP$-module is artinian if and only if every submodule has finite hollow dimension.

In [7], the authors defined and studied rings for which every (simple) cosingular module is projective (these rings are called (SCP) CP). They showed that for a ring R, every simple cosingular R-module is projective if and only if R is a GV (GCO) ring. They give some conditions for a ring to be CP. It is also shown for a right perfect ring R under an assumption that R is right CP if and only if R is a left and right artinian serial ring with $J(R)^2 = 0$.

Inspiring by [4] and [7], in this paper we will study rings (modules) R (M) such that every (M-)cosingular R-module (in $\sigma[M]$) is discrete. We call them CD-rings (CD-modules). The aim of this article is to characterize rings for which every cosingular R-module is discrete.

Sections 2 is devoted to study rings for which every cosingular module is discrete. We show that for a semilocal ring R, R is right CD if and only if $\frac{R}{\overline{Z}(R_R)}$ is semisimple. For a right perfect ring R, it is proved that every $\overline{Z}^2$-torsionfree R-module is (quasi-)discrete if and only if R is right CD. We also present some examples to illustrate different concepts.

2. CD-Modules and Rings

In this section we introduce a new class of modules (resp. rings) namely CD-modules (resp. rings). An R-module M is CD, provided that every M-cosingular R-module in $\sigma[M]$ is discrete. The class of CD-modules contains semisimple modules and V-modules. We introduce and study rings for which every cosingular module is discrete (in this case we call them CD-rings). We also investigate some general properties and some characterizations of CD-rings. For a ring R, we show that R is CD if and only if every cosingular module is semisimple in the case where the class of cosingular R-modules is closed under taking homomorphic images.

It is not hard to verify that a ring R is right CD if and only if the right R-module R_R is CD if and only if every cyclic right R-module is CD.

We consider the following conditions on a module M:

(D_0) For every decomposition $M = M_1 \oplus M_2$ of M, M_1 and M_2 are relatively projective;

(D_1) Every submodule of M lies above a direct summand of M;

(D_2) If $M/A \cong B \leq_\oplus M$, then $A \leq_\oplus M$;

(D_3) If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2 \leq_\oplus M$.

M is called discrete if it satisfies (D_1) and (D_2), quasi-discrete if it satisfies (D_1) and (D_3) and a lifting module if M satisfies (D_1).

We have the following hierarchy:

discrete $\Longrightarrow$ quasi-discrete $\Longrightarrow$ lifting $\Longrightarrow$ H-supplemented $\Longrightarrow$ $\oplus$-supplemented $\Longrightarrow$ supplemented

Proposition 2.1. *Any homomorphic images of a CD-module is CD. In particular, any direct summand of a CD-module is CD.*

As a consequence every ring homomorphic image of a CD-ring is CD.

Let $\mathcal{A}$ be a class of modules and $M \in \mathcal{A}$. We say that M is $\mathcal{A}$-projective in the case where M is projective relative to all elements of $\mathcal{A}$.

Theorem 2.2. *Let $\mathcal{A}$ be a class of modules in $Mod - R$ such that $\mathcal{A}$ is closed under finite direct sums. Consider the following conditions:*

(1) *Every module in $\mathcal{A}$ is semisimple;*

(2) *Every module in $\mathcal{A}$ is discrete;*

(3) *Every module in $\mathcal{A}$ is quasi-discrete;*

(4) *Every module in $\mathcal{A}$ satisfies* (D_0)*;*

(5) *Every module in $\mathcal{A}$ is $\mathcal{A}$-projective.*

Then, (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5). *If in addition, $\mathcal{A}$ is closed under homomorphic images, then* $(1) - (5)$ *are all equivalent.*

If we replace $\mathcal{A}$ with the class of cosingular modules, we have the following corollary:

Corollary 2.3. *Let R be a ring. If the class of cosingular right R-modules is closed under homomorphic images, then the following statements are equivalent:*

(1) *R is right CD;*

(2) *Every cosingular right R-module is semisimple;*

(3) *Every cyclic cosingular right R-module is semisimple;*

(4) *Every cosingular right R-module is quasi-discrete;*

(5) *Every cosingular right R-module satisfies* (D_0)*;*

(6) *Every cosingular right R-module is N-projective for every cosingular right R-module N.*

If one of above statements holds, every cosingular right R-module is self-projective.

Proposition 2.4. *Let R be a right CP-ring. Then the following statements are equivalent:*

(1) *R is right CD;*
(2) *Every cosingular right R-module is lifting;*
(3) *Every cosingular right R-module is H-supplemented;*
(4) *Every cosingular right R-module is $\oplus$-supplemented;*
(5) *Every cosingular right R-module is supplemented.*

Proposition 2.5. *Let R be a right perfect ring and M an R-module. Then the following are equivalent:*

(1) *Every direct product of M-projective right R-modules is quasi-discrete;*
(2) *Every direct product of M-projective right R-modules satisfies (D_0).*

In this case, the class of M-projective right R-modules is closed under direct products.

As a consequence of Proposition 2.5, we have a new characterization of commutative artinian rings.

Corollary 2.6. *Let R be a commutative perfect ring. Then the following are equivalent:*

(1) *R is artinian;*
(2) *Every direct product of projective R-modules is quasi-discrete;*
(3) *Every direct product of projective R-modules satisfies (D_0).*

Now we can replace $\mathcal{A}$ in Theorem 2.2 with the class of small modules.

Corollary 2.7. *Let R be a ring. Then the following statements are equivalent:*

(1) *Every small right R-module is semisimple;*
(2) *Every small right R-module is discrete;*
(3) *Every small right R-module is quasi-discrete;*
(4) *Every small right R-module satisfies (D_0);*
(5) *Every small right R-module is N-projective for every small right R-module N.*

The following introduces some basic properties of CD-modules.

Proposition 2.8. *Suppose that M is a CD-module. Then:*

(1) Every M-small module is semisimple. In particular every small submodule of M is semisimple.
(2) $Rad(M) \subseteq Soc(M)$.
(3) $Rad(M) \ll M$.
(4) If $M/\overline{Z}_M(M)$ is semisimple, then $Rad(M) \subseteq \overline{Z}_M(M)$.
(5) Every finitely generated submodule of $Rad(M)$ is artinian (noetherian).

By above Proposition, a CD-module can not be radical.

Recall from [11] that a ring R is a V-ring provided every simple R-module is injective, equivalently R is a V-ring if and only if every R-module has zero radical. Since the only cosingular module over a V-ring is zero, every V-ring is CD and CP. A ring R is *generalized co-semisimple* (GCO for short), provided that every simple singular module is injective. Note that R is right GCO if and only if R is right GV.

The following is taken from [7] showing that a semiperfect ring R, is CP if and only if R is GV (see [7, Remark 3.3 and Corollary 3.5]).

Proposition 2.9. *Let R be a ring such that every (finitely generated) cosingular module is amply supplemented. Then R is right (GV) GCO if and only if R is right CP. (In this case R is right CD and the class of (finitely generated) cosingular right R-modules is closed under taking homomorphic images.)*

Let R be a ring. Then R is right (resp. left) SI provided every singular right (resp. left) R-module is injective. These rings were introduced and fully investigated by Goodearl in [2].

Note that every semiperfect right SI-ring is right CD by Proposition 2.9.

Remark 2.10. *If for a CD-module M, the class of M-cosingular modules is closed under factor modules, then every M-cosingular M-injective module is zero. So for a right CD-ring R such that the class of cosingular right R-modules is closed under homomorphic images (e.g. semiperfect right SI-rings), every cosingular injective right R-module is zero. This answers that one of questions posed by Talebi and Vanaja (see* [8, Page 1460, Question 3]*).*

Proposition 2.11. *Let R be a right GV-ring. Then R is right CD if and only if every cyclic cosingular right R-module is amply supplemented.*

Let $\mathcal{S}'$ and $\mathcal{S}$ denote the classes of left and right small modules respectively. Recall from [8], $\overline{Z}({}_RR) = Rej(R, \mathcal{S}') = \bigcap\{Kerf \mid f : R \to U, U \in \mathcal{S}'\}$ and $\overline{Z}(R_R) = Rej(R, \mathcal{S}) = \bigcap\{Kerf \mid f : R \to U, U \in \mathcal{S}\}$. By [1, Corollary 8.23], $\overline{Z}({}_RR)$ and $\overline{Z}(R_R)$ are two-sided ideals of R. We say R is a *(left) right cosingular ring*, if ($\overline{Z}({}_RR) = 0$) $\overline{Z}(R_R) = 0$.

Remark 2.12. *Let R be a right cosingular right CD-ring. Then by Corollary 2.3, every cosingular right R-module is R-projective. In particular, any finitely generated cosingular right R-module is projective.*

Let M be a module. M is said to have *finite hollow dimension*, in case there exists an epimorphism $f : M \to \prod_{i=1}^{n} H_i$, with H_i is hollow and $Kerf \ll M$. In this case, we say the hollow dimension of M is n.

Proposition 2.13. *The following statements are equivalent for a CD module M with finite hollow dimension:*
(1) *M is finitely cogenerated;*
(2) *$Rad(M)$ is Artinian.*

Proposition 2.14. *For a right CD-ring R the following statements hold:*
(1) Every small right R-module is semisimple.
(2) $J(R) \subseteq Soc(R_R)$.
(3) $J(R)^2 = 0$.
(4) $R/\overline{Z}(R_R)$ is semiperfect.

Corollary 2.15. *Let R be a commutative domain. Then the following are equivalent:*
(1) *R is CD (CP);*
(2) *R is a field.*

Proposition 2.16. *Let R be a ring such that every finitely generated cosingular R-module is semisimple. If for every R-module M, $\overline{Z}(M) \leq_{\oplus} M$, then R is a CP-ring.*

Corollary 2.17. *R is a CP-ring in each of the following cases:*
(1) R is a CD-ring such that the class of cosingular R-modules is closed under factor modules and for every R-module M, $\overline{Z}(M) \leq_{\oplus} M$.
(2) Every R-module is a direct sum of a non-cosingular R-module and a semisimple R-module. (Clearly in this case R is also CD)

Note that by [7] for a right perfect ring R, every cyclic cosingular right R-module is projective if and only if R is right CP.

Recall from [3, p. 236], a ring R is a *right (left) good ring*, in case for every (left) right R-module M, we have $(JM = Rad(M))$ $MJ = Rad(M)$. Every semilocal ring (rings R for which $R/J(R)$ is semisimple) is a (left) right good ring. By a similar argument to [9, Corollary 2.7], over a semilocal ring R, we have $\overline{Z}(R_R) = Soc(_RR)$ and $\overline{Z}(_RR) = Soc(R_R)$.

The following introduces a large class of two-sided CD-rings.

Proposition 2.18. *Let R be a semilocal ring such that $(J(R) \subseteq Soc(_RR))$ $J(R) \subseteq Soc(R_R)$. Then R is (right) left CD. In particular, every semilocal ring with $J(R)^2 = 0$ is left and right CD.*

The following example introduces a right (left) cosingular semilocal ring which is not right (left) CD.

Example 2.19. *Let D be a commutative local integral domain with field of fractions Q (for example we might take D the localization of the integers Z by a prime number p, i.e. D is the subring of the field of rational numbers consisting of fractions a/b such that b is not divisible by p).*

Let $R = \begin{pmatrix} D & Q \\ 0 & Q \end{pmatrix}$. The operations are given by the ordinary matrix operations. Since D is local it has a unique maximal ideal, say m and the Jacobson radical of R is $J(R) = \begin{pmatrix} m & Q \\ 0 & 0 \end{pmatrix}$. Now, $R/J(R) \cong (D/m) \times Q$. Thus R is semilocal. On the other hand if we suppose that D has zero socle, then R has zero left socle. But on the other hand, R has non-zero right socle, namely $\overline{Z}(_RR) = Soc(R_R) = \begin{pmatrix} 0 & Q \\ 0 & Q \end{pmatrix}$, since the right R-module structure coincides for this submodule with the Q-vectorspace structure of $Q \times Q$. It follows that R is right cosingular but not left cosingular. Since $J(R) \nsubseteq Soc(R_R)$ and $J(R) \nsubseteq Soc(_RR)$, R is neither right CD (CP) nor left CD (CP).

The following example shows that the class of CD-rings contains properly the class of V-rings.

Example 2.20. *Let F be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ be the ring of 2×2 upper tringular matrices. It is well-known that R is a right and left (SI) GV-ring which is neither a right nor a left V-ring (because $J(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$). Since R is left and right artinian serial with $J(R)^2 = 0$, by Proposition 2.18, R is left and right CD.*

Let R be a ring. Then R is called right *Harada* (right H-ring for short) provided that every injective right R-module is lifting. It is well-known that R is right H-ring if and only if every right R-module decomposed to a small module and an injective module.

Proposition 2.21. *Let R be a right CD right H-ring. Then R is an (left and right) artinian serial ring with $J(R)^2 = 0$.*

Remark 2.22. *Note that a semilocal non-semisimple ring with $Soc(_RR)$ right semisimple can not be left and right CP. For if, assume that R is a semilocal left and right CP-ring. Then $J(R) \subseteq Soc(_RR) = \overline{Z}(R_R) \leq_{\oplus} R$. Since $J(R) \ll R$ and $\overline{Z}(R_R) \leq_{\oplus} R$, we have $J(R) \ll \overline{Z}(R_R)$. Since $\overline{Z}(R_R)$ is right semisimple, it follows that $J(R) = 0$. Hence R is semisimple. The ring $\frac{\mathbb{Z}}{4\mathbb{Z}}$ is a local CD-ring but not CP (GV).*

Proposition 2.23. *Let R be a ring such that the class of cosingular right R-modules is closed under homomorphic images. Then the following statements are equivalent:*
(1) *R is right CD;*
(2) *Every cosingular right R-module is semisimple;*
(3) *The ring $R/\overline{Z}(R_R)$ is semisimple.*

A ring R is right (resp. left) SC, if every right (resp. left) singular R-module is continuous (see [6]). Also module theoretic version of SC-rings can be found in [10]. A module M is SC if every M-singular module is continuous.

Theorem 2.24. *The following statements are equivalent for a semilocal ring R:*
(1) *R is a left SC-ring with $Soc(_RR) \leq_e R$;*
(2) *R is right CD;*
(3) *The ring $R/\overline{Z}(R_R)$ is semisimple.*
If R satisfies one of these conditions, then R is a left CD-ring.

Corollary 2.25. *Let R be a commutative semilocal ring. Then R is CD if and only if R is SC.*

Remark 2.26. *Every non-trivial ideal of a local CD-ring (CP-ring) R is semisimple. However, R need not be semisimple. The ring $R = \mathbb{Z}/4\mathbb{Z}$ is a local CD-ring by Theorem 2.24, that its only ideal is simple and R is not semisimple.*

Recall that a ring R is (left) right nonsingular, if $(Z_l(R) = \{x \in R \mid Ix = 0, I \leq_e (_RR)\} = 0)$ $Z_r(R) = \{x \in R \mid xI = 0, I \leq_e (R_R)\} = 0$.

Proposition 2.27. *A ring R is left nonsingular, semilocal with $R/\overline{Z}(R_R)$ semisimple if and only if R is semisimple.*

The following example shows that a CD-ring need not be CP (SI or GV).

Example 2.28. *Let p and q be two distinct prime numbers. Then for $m, n \in \{0, 1, 2\}$, the ring $R = \frac{\mathbb{Z}}{p^m q^n \mathbb{Z}}$ is a CD-ring but not CP (m and n both can not be zero and also both can not be one).*

Definition 2.29. *We call a module M, $\overline{Z}^2$-torsionfree, in case $\overline{Z}^2(M) = 0$. It is easy to see that every cosingular module is $\overline{Z}^2$-torsionfree.*

The class of $\overline{Z}^2$-torsionfree modules is closed under submodules, direct sums and direct products (see [8, Proposition 2.1]). It is also followed by [5, Theorem 4.41] and [8, Proposition 2.1 and Theorem 3.5] that for a perfect ring R, the class of $\overline{Z}^2$-torsionfree R-modules is also closed under factor modules.

Theorem 2.30. *Let R be a right perfect ring. Consider the following conditions:*
(1) Every $\overline{Z}^2$-torsionfree right R-module is discrete;
(2) Every $\overline{Z}^2$-torsionfree right R-module is quasi-discrete;
(3) R is right CD.
Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3). If R is right GV, then (3) $\Longrightarrow$ (1).

Let R be a ring such that every cyclic cosingular R-module is discrete. Then R need not be a CD-ring as the following shows.

Example 2.31. *The ring $R = \mathbb{Z}_8$ is a local ring such that $\frac{R}{\overline{Z}(R)} = \frac{R}{Soc(R)}$ is not semisimple. So by Theorem 2.24, R is not a CD-ring. Let M be a nonzero cyclic R-module. Then M is isomorphic to $M_1 = \frac{R}{(2)} = \frac{R}{J(R)}$ or $M_2 = \frac{R}{(4)} = \frac{R}{Soc(R)}$ or $M_3 = R$. The module M_1 is simple. The module M_2 is an indecomposable local R-module and M_3 is discrete since R is semiperfect. Hence all cyclic (cosingular) R-modules are discrete.*

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A DETAILED STUDY OF HANKEL AND SUB-HANKEL'S POLAR IDEALS

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Abstract. The theme of this work is the theory of homaloidal determinants. The focus is on the homological properties of the determinant of a generic Hankel matrix and one of its degenerations as a method of studying their homaloidal behavior. In characteristic zero we show that the first has nonvanishing Hessian (hence its polar map defines a dominant rational map) but it is non-homaloidal. The case of the degenerate determinant has been proved to be homaloidal by C. Ciliberto, F. Russo and A. Simis (Advances in Math.**218**:1759–1805, 2008). We determine the ideal theoretic and numerical invariants of the corresponding gradient (polar) ideal, as well as its homological nature. These can in turn be used to simplify a few passages in the proofs of Ciliberto-Russo-Simis. All results draw on some nontrivial underlying commutative algebra and the nature of its use is one of the assets of this work.

This is a joint work with Aron Simis‡.

1. Introduction

The subject of Cremona transformations is a classical chapter of algebraic geometry. However, the classification of such maps in projective space $\mathbb{P}^n$ is well understood only for $n \leq 2$. In higher dimension the structure of the Cremona group is far from being fully understood, and classification is poorly known. An important class of Cremona maps of $\mathbb{P}^n$ consists of the so-called *polar maps*, i.e., rational maps $\nabla f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ defined by the partial derivatives of a form $f \in R := k[\mathbf{x}] = k[x_0, \ldots, x_n]$ over a field of characteristic zero. The hypersurface $V(f)$ is called a *homaloidal hypersurface* if ∇f is a Cremona transformation of $\mathbb{P}^n$, in which case, by extension, f is also dubbed a *homaloidal polynomial*. The simplest example of a homaloidal hypersurface $V(f)$ is a smooth quadric: in this case the polar map ∇f is an invertible linear map. This is also the only case of a reduced homaloidal polynomial if $n = 1$.

The plane case of homaloidal hypersurfaces has been settled by Dolgachev in [2]:

Theorem. A reduced homaloidal curve $V(f) \subset \mathbb{P}^2$ has degree ≤ 3.

From this it is fairly easy to show that the curve is one of the following:

(i) A smooth conic;
(ii) The union of three distinct non-concurrent lines;

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(iii) The union of a smooth conic with one of its tangent lines.

This result motivated experts to ask whether an analogue would hold for higher dimensions.

Question 1.1. Let $f \in k[\mathbf{x}] = k[x_0, \ldots, x_n]$ denote a squarefree form. If f is homaloidal when is $\deg(f) \leq n+1$?

This question has been negatively settled in [1, Theorem 3.13], by showing that for any $n \geq 3$ and any $d \geq 2n-3$ there exist irreducible homaloidal hypersurfaces of degree d.

Our approach is algebraic throughout. Needless to justify, we will assume throughout that the base field has characteristic zero. From the geometric point of view, the study of a polar map in characteristic zero drives primevally through the properties of the Hessian determinant $h(f)$ of f, the reason being the classically known criterion for the dominance of the polar map in terms of the non vanishing of the corresponding Hessian determinant. Although this criterion admits a vast generalization to arbitrary rational maps in terms of the Jacobian determinant of a basis of the corresponding linear system, it is in the polar case that the notion takes full role.

Here is a brief description of the contents of the paper.

2. Section 2

In section 2 we initially review a few basic notions of ideal theory to be used throughout. These are mainly related to the syzygies of an ideal and its main associated algebras, stressing some or other notation that may not be universally accepted. Next there are a some preliminaries about homaloidal polynomials and displays some elementary examples. A discussion of birationality criteria is annotated for the reader's convenience. Our source for these criteria is generally [3], where the language looks quite encompassing both for the algebraist and the geometer.

Sections 3 and 4 contain the core of the results and each is subdivided in several subsections.

3. Sections 3

In the section 3 the first consideration is the generic Hankel matrix of arbitrary size. Thus, let $\mathcal{H}_m$ stand for the generic $m \times m$ Hankel matrix and let $P \subset R$ denote its ideal of submaximal minors (i.e., $(m-1)$-minors). We prove that P is the minimal primary component of the gradient ideal $J = J(f) \subset R$ of $f := \det \mathcal{H}_m$. Although this assertion sounds naturally guessed, not so much its proof. We have used a bouquet of arguments, ranging from multiplicities to initial ideals to Plücker relations (straightening laws) of Hankel maximal minors via the Gruson–Peskine change of matrix trick.

From this it is but one step to guess that the only remaining associated (necessarily embedded) prime of R/J is the obvious candidate $Q := I_{m-2}(\mathcal{H}_m)$ – defining the singular locus of the determinantal variety $V(P)$. This guess is equivalent to the expected equality $J : P = Q$. We chose to include the latter guess in a conjecture of much larger scope to the effect that

$$JP^i : P^{i+1} = I_{m-2-i}(\mathcal{H}_m),$$

for $0 \leq i \leq m-2$. An almost immediate consequence of this conjectured statement is that J is a minimal reduction of P with reduction number $m-2$. This gives quite a spectacular relationship of algebraic content between these two ideals.

By the result proved in [4, Theorem 3.3.5] to the effect that the *linear syzygy rank* of J is $3 < 2m-2$ for $m \geq 3$ we get f is not homaloidal if J is of linear type .

In the case where $m = 3$ we are able to solve all previous conjectures in the affirmative.

4. Section 4

In section 4 our focus is on certain degenerations of the generic versions of the previous section. We avoid calling them specializations since the numerical invariants of the various ideals in consideration may change as do their properties of interest to this work. Actually, the only sort of degeneration introduced here is by means of replacing some variables (entries) in strategic positions by zeros.

The examples show that degenerating can both destroy or give rise to the property of homaloidness. This phenomenon is yet to be understood from the geometric point of view, but even the underlying algebra is not cleat either.

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DUAL OF CODES OVER FINITE QUOTIENTS OF POLYNOMIAL RINGS

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ABSTRACT. Let $A = \frac{\mathbb{F}[x]}{f(x)}$, where $f(x)$ is a monic polynomial over a finite field $\mathbb{F}$. In this talk, we study the relation between A-codes and their duals. In particular, we state a counterexample and a correction to a result of Berger and Amrani (Codes over finite quotients of polynomial rings, Finite Fields Appl.,2014) and present an efficient algorithm to find a system of generators for the dual of a given A-code.

1. Introduction

Throughout this paper $A = \frac{\mathbb{F}[x]}{f(x)}$, where $f(x)$ is a monic polynomial over a finite field $\mathbb{F}$. Moreover, $\deg(f) = m$ and $|\mathbb{F}| = q$. We consider elements of A as polynomials of degree $< m$ where the arithmetic is done modulo $f(x)$. By an A-code of length l we mean an A-submodule of A^l.

In the case $f(x) = x^m - 1$ and $l = 1$, A-codes are the well-known cyclic codes. Also if $l > 1$ with $f(x) = x^m - 1$, then A-codes represent quasi-cyclic codes which have recently gained a great attention (see, for example [2, 3]). In [2], a canonical generator matrix for quasi-cyclic codes is given. In [1] these results are generalized to A-codes.

Assume that $0 \neq C$ is an A-code of length l and $u = (u_1(x), u_2(x), \ldots, u_l(x)) \in A^l$. The *leading index* of u, denoted $L_{\text{ind}}(u)$ is the smallest integer i such that $u_i \neq 0$ and $L_{\text{coef}}(u) = u_{L_{\text{ind}}(u)}$ is called the *leading coefficient* of u. Also by $L_{\text{ind}}(C)$ we mean $\min\{L_{\text{ind}}(u) | u \in C\}$ and $L_{\text{coef}}(C)$ is the single monic polynomial $g(x)$ with the minimum degree such that there is a $c \in C$ with $L_{\text{ind}}(c) = L_{\text{ind}}(C)$ and $L_{\text{coef}}(c) = g(x)$. An element $c \in C$ satisfying this condition is called a *leading element* of C.

Recursively set $C^{(1)} = C$ and if $C^{(n)} \neq \{0\}$ then $C^{(n+1)} = \{c \in C^{(n)} | L_{\text{ind}}(c) > L_{\text{ind}}(C^{(n)})\}$ (if this set is empty, we set $C^{(n+1)} = \{0\}$). Let k be largest integer such that $C^{(k)} \neq \{0\}$ and assume that for $1 \leq j \leq k$, $g^{(j)}$ is a leading element of $C^{(j)}$. Then by Theorem 1 and Proposition 2 of [1], C is generated by $B = (g^{(1)}, \ldots, g^{(k)})$ (as an A-module) and k and $\deg(L_{\text{coef}}(g^{(i)})$'s are independent of the choice of $g^{(i)}$'s. Also $|C| = q^\alpha$ where $\alpha = km - \sum_{i=1}^{k} \deg(L_{\text{coef}}(g^{(i)}))$. Any B as above is called a *basis of divisors* of C.

Now set G to be the matrix whose i'th row is $g^{(i)}$. Suppose that g_{i,j_i} is the leading coefficient of the i'th row of G. If G has the property that $\deg(g_{t,j_i}) < \deg(g_{i,j_i})$ for all $1 \leq i \leq k$ and $t < i$, then G is called the *canonical generator matrix* of C and B

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is called the *canonical basis of divisors* of C. In [1, Theorem 2] it is shown that every A-code has a unique canonical generator matrix.

Let $C^{\perp} = \{(a_1, \ldots, a_l) \in A^l | \forall c \in C \quad \sum_{i=1}^{l} a_i c_i = 0\}$ be the dual of an A-code C. Section 2.6 of [1] states how to compute a system of generators of $C^{\perp}$. In this talk, we will show that the main theorem of [1, Section 2.6] is not correct and through a series of lemmas we state a correction of this theorem. Also we present an efficient algorithm to find a generator matrix for $C^{\perp}$ (that is, a matrix, rows of which generate $C^{\perp}$ as an A-module).

2. Main Results

In the sequel, we assume that C is an A-code of length l and that $g^{(1)} = (g_{1,1}(x), \ldots, g_{1,l}(x))$ is the first element of its canonical basis of divisors. Also we set C' to be the punctured code of $C^{(2)}$ on the first position and assume that G' is the canonical generator matrix of C'. The following theorem is is claimed to be proved in [1].

Incorrect Theorem 2.1 ([1, Theorem 3])**.** *Suppose that* $L_{\mathrm{ind}}(C) = 1$ *and* $h_{1,1}(x) = \frac{f(x)}{g_{1,1}(x)} \pmod{\mathrm{f(x)}}$. *Let* H' *be a generator matrix of* $C'^{\perp}$. *Then*

$$H = \left(\begin{array}{c|c} 1 & 0 \ldots 0 \\ \hline 0 & \\ \vdots & H' \\ 0 & \end{array} \right) \times \begin{pmatrix} h_{1,1} & 0 & \ldots & \ldots & 0 \\ -g_{1,2} & g_{1,1} & \ddots & \ddots & \vdots \\ -g_{1,3} & 0 & g_{1,1} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -g_{1,l} & 0 & \ldots & 0 & g_{1,1} \end{pmatrix}$$

is a generator matrix for $C^{\perp}$.

To present a counterexample of 2.1, we need the following result.

Proposition 2.2. *Let* G *be a* $k \times l$ *generator matrix for an* A*-code* C. *Suppose that* $g^{(i)}$ *is the* i*'th row of* G *and set* $h_i(x) = \frac{f(x)}{L_{\mathrm{coef}}(g^{(i)})}$. *Then* $(g^{(1)}, \ldots, g^{(k)})$ *is a basis of divisors of* C *if and only if the following hold.*

- **(i)** G *is in echelon form.*
- **(ii)** $L_{\mathrm{coef}}\left(g^{(i)}\right) | f(x)$.
- **(iii)** $h_i g^{(i)}$ *is a linear combination of rows* $i+1, \ldots, k$ *of* G.

Example 2.3. *Let* $\mathbb{F} = \mathbb{F}_2, f(x) = x^2(x^3+1)$ *and* C *be the* A*-code of length 3 which is generated by*

$$G = \begin{pmatrix} x & x & 0 \\ 0 & x^2 & 1 \\ 0 & 0 & x^3+1 \end{pmatrix}.$$

Using 2.2, we can see that G *is the canonical generator matrix of* C. *Also* C' *is generated by* $(x^2, 1)$ *and* $(0, x^3+1)$. *One can readily check that a generator matrix for* $C'^{\perp}$ *is* $H' = (1, x^2)$. *Thus if* H *is as in 2.1, then*

$$H = \begin{pmatrix} x(x^3+1) & 0 & 0 \\ x & x & x^3 \end{pmatrix}.$$

Clearly $u = (1, 1, x^2) \in C^{\perp}$*. But if* u *is a linear combination of the rows of* H*, then for some* $\alpha, \beta \in \mathbb{F}[x]$ *we have* $\alpha x(x^3 + 1) + \beta x \equiv 1 \pmod{\mathrm{f(x)}}$ *which leads to* $x|1$*, a contradiction. Thus* H *is not a generator matrix of* $C^{\perp}$ *and 2.1 is not correct.*

To present the correct generator matrix for $C^{\perp}$, we need the following lemmas.

Lemma 2.4. *For each* $c' \in C'^{\perp}$ *there is a* $c_1 \in A$ *such that* $(c_1|c')$ *(the concatenation of* c_1 *to* c'*) is an element of* $C^{\perp}$*.*

Lemma 2.5. *Suppose that* $L_{\mathrm{ind}}(C) = 1$ *and* $h_{1,1}$ *and* $H' = (h'_{ij})_{2 \le i \le k}^{2 \le j \le l}$ *are as in 2.1. Then* $g_{1,1} | \sum_{j=2}^{l} h'_{ij} g_{1j}$ *for each* $2 \le i \le k$*.*

Lemma 2.6. *Let* $c' \in C'^{\perp}$ *and* $c_1 \in A$ *be such that* $(c_1|c') \in C^{\perp}$*. Also suppose that* $L_{\mathrm{ind}}(C) = 1$ *and* $h_{1,1}$ *and* H' *are as in 2.5. Set* $h'^{(i)}$ *to be the i'th row of* H' *and*

$$\alpha_i = -\frac{\sum_{j=2}^{l} h'_{ij} g_{1j}}{g_{1,1}} \pmod{\mathrm{h}_{1,1}}.$$

If $c' = \sum_{i=2}^{k} \lambda_i h'^{(i)}$ *where* λ_i*'s are in* A*, then* $h_{1,1} \Big| \left(c_1 - \sum_{i=2}^{k} \lambda_i \alpha_i\right)$*.*

Theorem 2.7. *Assuming that* $L_{\mathrm{ind}}(C) = 1$ *and* $l > 1$ *and using the notations of 2.6, a generator matrix for* $C^{\perp}$ *is*

$$H = \left(\begin{array}{c|c} h_{1,1} & 0 \ldots 0 \\ \hline \alpha_1 & \\ \alpha_2 & \\ \vdots & H' \\ \alpha_k & \end{array}\right).$$

It should be noted that the above theorem is correct when $C' = 0$, in which case $H' = I_{l-1 \times l-1}$. Also if $l = 1$, then clearly $H = (h_{1,1})$ is the generator matrix of $C^{\perp}$.

Using 2.7 we get the following efficient recursive algorithm for computing a generator matrix of $C^{\perp}$. The matrix generated by this algorithm is not in the canonical form. But since we calculated α_i's modulo $h_{1,1}(x)$ instead of $f(x)$ in 2.6, this matrix is very similar to the canonical form — just we should delete the zero rows and then look at the rows and columns in the reverse order. More concretely:

Theorem 2.8. *Let* $H_{k \times l}$ *be the generator matrix of* $C^{\perp}$ *calculated by Algorithm 1 after deleting the possible zero rows. Let* H^R *be the* $k \times l$ *matrix with* $h^R_{ij} = h_{k-i+1, l-j+1}$*. Then* H^R *is the canonical generator matrix of* $C^{\perp R}$ *the reciprocal dual of* C *(that is,* $\{(c_l, \ldots, c_1) | (c_1, \ldots, c_l) \in C^{\perp}\}$*).*

Note that $C^{\perp}$ and $C^{\perp R}$ are equivalent codes. So by finding parameters and properties of one of these codes, we have found those of the other one. As an example of usage of 2.7 we state the following.

Corollary 2.9. *Suppose that* C *has a basis of divisors consisting of one element, say* $(g_1(x), g_2(x), \ldots g_l(x))$*. Then* C *is self-dual if and only if either* $l = 1$ *and* $f(x) = (g_1(x))^2$ *in* $\mathbb{F}[x]$ *or* $l = 2$*,* $g_1 = 1$ *and* $g_2^2 = -1$ *in* A*.*

At the end, we notice a property of A-codes which is important in proving the above results. This property is that $|C| \times |C^{\perp}| = |A|^l$ (see Properties below [1, Definition 2]). Combining this with [4, Theorem 3.5] we immediately get the following result.

Algorithm 1 gen-mat-dual(G) (Calculates a generator matrix of dual of an A-code C)

Input: A canonical generator matrix $G_{k\times l}$ of C
Output: A generator matrix H of $C^{\perp}$

if the first column of G is zero **then**
  if l=1 **then**
    return $H = (1)$
  else
    set G' to be G with the first column deleted
    H' =gen-mat-dual(G')
    return $H = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & H' \end{array}\right)$
  end if
else
  if $l = 1$ **then**
    return $H = \left(\frac{f(x)}{g_{1,1}(x)} \pmod{\mathrm{f(x)}}\right)$
  else
    let G' be G with the first row and column deleted
    H' =gen-mat-dual(G')
    construct and return H as in 2.7.
  end if
end if

Proposition 2.10. *If $A = \frac{\mathbb{F}[x]}{f(x)}$ where $f(x)$ is a monic polynomial over a finite field $\mathbb{F}$ then the number of maximal ideals and the number of minimal ideals of A are the same. Also the MacWilliams identity holds for every A-code.*

It should be noted that the MacWilliams identity is a relation between the weight enumerator of a linear code and that of its dual (see, for example [5, Section 5.2]). This relation is well-known for codes over finite fields and is proved to hold for codes over many other rings. In particular, in [4] it is shown that a generalization of this relation holds for linear codes over finite commutative rings which have the same number of maximal and minimal ideals.

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COHEN-MACAULAY AND GORENSTEINNESS OF $A \bowtie^f J$

PARVIZ SAHANDI[†]

Abstract. Let A and B be commutative rings with unity, $f : A \to B$ a ring homomorphism and J an ideal of B. Then the subring $A \bowtie^f J := \{(a, f(a)+j) | a \in A$ and $j \in J\}$ of $A \times B$ is called the amalgamation of A with B along with J with respect to f. In this talk, among other things, we investigate the Cohen-Macaulay and (quasi-)Gorenstein properties of the ring $A \bowtie^f J$.

This is a joint work with N. Shirmohammadi[‡] and S. Sohrabi[††].

1. Introduction

In [1], D'Anna, Finocchiaro, and Fontana have introduced the following new ring construction. Let A and B be commutative rings with unity, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. They introduced the following subring

$$A \bowtie^f J := \{(a, f(a) + j) | a \in A \text{ and } j \in J\}$$

of $A \times B$, called the *amalgamation of A with B along J with respect to f*. This construction generalizes the amalgamated duplication of a ring along an ideal (introduced and studied in [2]). Moreover, several classical constructions such as the Nagata's idealization (cf. [3, page 2]), the $A + XB[X]$ and the $A + XB[[X]]$ constructions can be studied as particular cases of this new construction (see [1, Examples 2.5 and 2.6]).

In this paper we will study when the amalgamated algebra $A \bowtie^f J$ is Cohen-Macaulay or (quasi-)Gorenstein.

2. Cohen-Macaulay property of $A \bowtie^f J$

Let us fix some notation which we shall use frequently throughout this section: A, B are two commutative rings with unity, A is Noetherian, $f : A \to B$ is a ring homomorphism, and J denotes an ideal of B, such that $A \bowtie^f J$ is Noetherian.

To state the main result we need to introduce some terminology. Let $(A, \mathfrak{m})$ be a Noetherian local ring, and N an A-module (not necessarily finitely generated). The *depth* of N over A is defined by the non-vanishing of the local cohomology modules $H^i_{\mathfrak{m}}(N)$, with respect to $\mathfrak{m}$, in the way that

$$\operatorname{depth} N := \operatorname{depth}_A N := \inf\{i | H^i_{\mathfrak{m}}(N) \neq 0\}.$$

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An A-module N is said to be *big Cohen-Macaulay* if $\operatorname{depth}_A N = \dim A$. In the sequel, $\operatorname{Jac}(B)$ will denote the Jacobson radical of B.

Lemma 2.1. *Let $(A, \mathfrak{m})$ be a local ring, and $J \subseteq \operatorname{Jac}(B)$ be an ideal of B such that $f^{-1}(\mathfrak{q}) \neq \mathfrak{m}$, for each $\mathfrak{q} \in Spec(B) \backslash V(J)$. Then $dim A \bowtie^f J = dim A$, and $depth A \bowtie^f J = \min\{depth A, depth_A J\}$.*

Theorem 2.2. *With the assumptions of Lemma 2.1, the following statements hold.*

(1) *If $A \bowtie^f J$ is Cohen-Macaulay, then so does A.*
(2) *Further assume that $depth J < \infty$. Then $A \bowtie^f J$ is Cohen-Macaulay if and only if A is Cohen-Macaulay and J is a big Cohen-Macaulay module.*

Corollary 2.3. *Let $(A, \mathfrak{m})$ be a local ring, and $J \subseteq \operatorname{Jac}(B)$ be an ideal of B such that J is a finitely generated A-module. Then $A \bowtie^f J$ is Cohen-Macaulay if and only if A is Cohen-Macaulay and J is a maximal Cohen-Macaulay A-module.*

A finitely generated module M over a Noetherian ring A satisfies Serre's condition (S_n) if $\operatorname{depth} M_{\mathfrak{p}} \geq \min\{n, \dim M_{\mathfrak{p}}\}$, for all $\mathfrak{p} \in \operatorname{Spec}(A)$.

Corollary 2.4. *The following statements hold.*

(1) *Assume that for each $\mathfrak{p} \in V(f^{-1}(J))$ and each $\mathfrak{q} \in Spec(B) \backslash V(J)$, $f^{-1}(\mathfrak{q}) \neq \mathfrak{p}$ (e.g. if f is surjective or J is a nil ideal of B). If $A \bowtie^f J$ satisfies (S_n), then so does A.*
(2) *Assume that $J^2 = 0$ and that J is a finitely generated A-module. If $A \bowtie^f J$ satisfies (S_n), then so does J.*

One can employ Corollary 2.4 to deduce that the property (S_n) is retained under the trivial extension construction.

Corollary 2.5. *Let $(A, \mathfrak{m})$ be a local ring, and M be a finitely generated A-module. Then the trivial extension $A \ltimes M$ satisfies (S_n) if and only if A and M satisfy (S_n).*

Let us recall that a finitely generated module M over a Noetherian local ring $(A, \mathfrak{m})$ is said to be a *generalized Cohen-Macaulay* A-module if $H^i_{\mathfrak{m}}(M)$ is of finite length for all $i < \dim M$.

Theorem 2.6. *Let $(A, \mathfrak{m})$ be a local ring, and $J \subseteq \operatorname{Jac}(B)$ be an ideal of B such that J is a finitely generated A-module. Then $A \bowtie^f J$ is a generalized Cohen-Macaulay ring if and only if A and J are generalized Cohen-Macaulay and $dim J \in \{0, dim A\}$.*

3. Gorenstein property of $A \bowtie^f J$

We now outline to recall some terminology. An A-module K is called a canonical module of A if

$$K \otimes_A \widehat{A} \cong \operatorname{Hom}_A(H^{\dim A}_{\mathfrak{m}}(A), E_A(A/\mathfrak{m})),$$

where $\widehat{A}$ is the $\mathfrak{m}$-adic completion of A, and $E_A(A/\mathfrak{m})$ is the injective hull of $A/\mathfrak{m}$ over A. We say that A is a *quasi-Gorenstein ring* if a canonical module of A exists and it is a free A-module (of rank one).

Theorem 3.1. *Assume that A is Noetherian local, and $J \subseteq \operatorname{Jac}(B)$, which is finitely generated A-module. The following statements hold.*

(1) *If $\widehat{A}$ satisfies (S_2) and J is a canonical module of A, then $A \bowtie^f J$ is quasi-Gorenstein.*
(2) *Assume that $J^2 = 0$ and that $Ann_A(J) = 0$. If $A \bowtie^f J$ is quasi-Gorenstein, then A satisfies (S_2) and J is a canonical module of A.*
(3) *Assume that $Ann_{f(A)+J}(J) = 0$ and that $A \bowtie^f J$ is quasi-Gorenstein. Then $f^{-1}(J)$ is a canonical module of A. Further assume that f is surjective. Then $\widehat{A}$ satisfies (S_2).*
(4) *Assume that J is a flat A-module. If $A \bowtie^f J$ is quasi-Gorenstein, then A is quasi-Gorenstein.*

Corollary 3.2. *Keep the assumptions of Theorem 3.1. Assume that A is Cohen-Macaulay and J is a canonical module of A. Then $A \bowtie^f J$ is Gorenstein.*

Corollary 3.3. *Keep the assumptions of Theorem 3.1. Assume that at least one of the following conditions holds*

(1) *f is an isomorphism and $Ann_B(J) = 0$; or*
(2) *$J^2 = 0$ and $Ann_A(J) = 0$.*

Then $A \bowtie^f J$ is Gorenstein if and only if A is Cohen-Macaulay and J is a canonical module of A.

Corollary 3.4. *Keep the assumptions of Theorem 3.1 and assume that $Ann_{f(A)+J}(J) = 0$. If $A \bowtie^f J$ is Gorenstein, then A is Cohen-Macaulay and $f^{-1}(J)$ is a canonical ideal of A.*

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PSEUDO-SUPPORT AND COFINITENESS OF LOCAL COHOMOLOGY MODULES

MONIREH SEDGHI

ABSTRACT. Let M denote a non-zero finitely generated module over a commutative Noetherian local ring $(R,\mathfrak{m})$ and let I be an ideal of R. In this talk it is shown that if R is complete, then for each fixed integer $s \geq 1$ the local cohomology module $H^s_{\mathfrak{m}}(M)$ is I-cofinite precisely when $\mathrm{Psupp}^s(M) \cap V(I) \subseteq V(\mathfrak{m})$, where $\mathrm{Psupp}^s(M) = \{\mathfrak{p} \in \mathrm{Spec}\, R \,|\, H^{s-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0\}$ denotes the *s-th pseudo-support* of M. In particular, we show that $H^s_{\mathfrak{m}}(M)$ is finitely generated if and only if $\mathrm{Psupp}^s(M) \subseteq V(\mathfrak{m})$. Also, we show that if $J \subseteq I$ are one dimensional ideals of R, then $H^t_I(M)$ is J-cominimax and $H^t_I(M)$ is finitely generated (resp. minimax) if and only if $H^t_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}})$ is finitely generated for all $\mathfrak{p} \in \mathrm{Spec}R$ (resp. $\mathfrak{p} \in \mathrm{Spec}R \setminus \mathrm{Max}R$).

1. INTRODUCTION

Throughout this paper, let R denote a commutative Noetherian ring (with identity) and I an ideal of R. For an R-module M, the i^{th} local cohomology module of M with respect to I is defined as

$$H^i_I(M) = \varinjlim_{n \geq 1} \mathrm{Ext}^i_R(R/I^n, M).$$

We refer the reader to [8] or [5] for more details about local cohomology. It is well known (see [5]) that, the i^{th} local cohomology module $H^i_I(M)$ vanishes unless $\mathrm{depth}(I,M) \leq i \leq \dim M$. An open problem in commutative algebra is to find a satisfactory criterion for $H^i_I(M)$ to be finitely generated (see [10, Problem 2]). The local cohomology modules are not finitely generated when $j \neq 0$ is the least integer such that $H^i_I(M) = 0$ for all $i > j$ (see [12, Proposition 3.1]). The problem of finding the least value of j for which $H^j_I(M)$ is not finitely generated has been investigated by Faltings in [6, 7] and Raghavan in [11]. Faltings showed, under some mild conditions on the ring, that $s(I,M) := \min\{\mathrm{depth} M_{\mathfrak{p}} + \mathrm{height}(I+\mathfrak{p})/\mathfrak{p} \,|\, I \nsubseteq \mathfrak{p} \in \mathrm{Supp} M\}$ is the least value of i for which $H^i_I(M)$ is not finitely generated (see [6]). Brodmann [2] has generalized this to give a criterion for when $H^i_I(M)$ is annihilated by a power of some ideal J contained in I. Finally Raghavan [11] showed that the power of J in Brodmann's theorem needed to annihilate $H^i_I(M)$ is independent of both I and J, depending only on the module M. In [10, Problem 3.3] Huneke asked the following wild problem: Is $0 \leq n \notin W$ if and only if $H^n_I(M)$ is finitely generated, where $W = \{\mathrm{depth} M_{\mathfrak{p}} + \mathrm{height}(I+\mathfrak{p})/\mathfrak{p} \,|\, I \nsubseteq \mathfrak{p} \in \mathrm{Supp}\, M\}$. This problem is not well understood. In fact, there are counterexamples to this conjecture given by Yosida in [12] and by Brodmann and Sharp in [4]. However, we are able to obtain a similar result to

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this problem whenever R is a complete local ring. More precisely we prove that if $(R,\mathfrak{m})$ is a complete local ring, M is finitely generated and n is a natural number, then the local cohomology module $H^n_{\mathfrak{m}}(M)$ is finitely generated precisely when $\mathrm{Psupp}^n(M) \subseteq V(\mathfrak{m})$. Here $\mathrm{Psupp}^s(M) = \{\mathfrak{p} \in \mathrm{Spec}\, R \,|\, H^{s-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0\}$ denotes the *s-th pseudo-support* of M (cf. [3]).

2. THE RESULTS

Before bringing the main theorem recall that if M is a finitely generated R-module, then the *i-th pseudo-support* $\mathrm{Psupp}^i(M)$ of M we define as (cf. [3]):

$$\mathrm{Psupp}^i(M) := \{\mathfrak{p} \in \mathrm{Spec}(R) \;:\; H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0\}.$$

Proposition 2.1. *Suppose that $(R,\mathfrak{m})$ is local and M a finitely generated R-module. Let $i \geq 1$ and I be an ideal of R such that $\mathrm{Psupp}^i(M) \cap V(I) \not\subseteq V(\mathfrak{m})$. Then the R-module $H^i_{\mathfrak{m}}(M)$ is not I-cofinite.*

Theorem 2.2. *Suppose that $(R,\mathfrak{m})$ is complete local, M is a finitely generated and n is a natural number. Then the local cohomology module $H^n_{\mathfrak{m}}(M)$ is finitely generated precisely when $\mathrm{Psupp}^n(M) \subseteq V(\mathfrak{m})$.*

Theorem 2.3. *Let I be an ideal of R such that $\dim R/I = 1$. Let M be a finitely generated R-module and $t \geq 0$ an integer. Then the following conditions are equivalent:*

(i) *$H^t_I(M)$ is a minimax R-module;*

(ii) *there exists an integer $n \geq 1$ such that $I^n H^t_I(M)$ is Artinian;*

(iii) *$H^t_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}})$ is a finitely generated $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in \mathrm{Spec}(R) \setminus \mathrm{Max}(R)$.*

Proposition 2.4. *Let $(R,\mathfrak{m})$ be local and M a non-zero finitely generated R-module. Let $J \subseteq I$ be ideals of R such that $\dim R/I = \dim R/J = 1$. Then $H^i_I(M)$ is J-cominimax, for all $i \geq 0$.*

Corollary 2.5. *Let $(R,\mathfrak{m})$ be local and M a non-zero finitely generated R-module. Let $J \subseteq I$ be ideals of R such that $\dim R/I = \dim R/J = 1$. Then the following hold:*

(i) *For all $i < f^J_I(M)$ the R-module $H^i_I(M)$ is minimax.*

(ii) *If there exists $\mathfrak{p} \in \mathrm{mAss}_R R/I$ such that $\dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} > 0$, then $f^J_I(M) < \infty$.*

Recall that an R-module N is said to be a *minimax module*, if there is a finitely generated submodule L of N, such that N/L is Artinian (cf. [13]). Also, for an ideal $\mathfrak{a}$ of R, an R-module M is said to be $\mathfrak{a}$-*cofinite* if M has support in $V(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a},M)$ is finitely generated for each i (cf. [9]). Finally, we say that M is $\mathfrak{a}$-*cominimax* if the support of M is contained in $V(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a},M)$ is minimax for all $i \geq 0$ (see [1]).

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BETTI NUMBERS, REGULARITY AND PROJECTIVE DIMENSION OF LOLLIPOP GRAPHS

SEYYEDE MASOOME SEYYEDI†

ABSTRACT. The lollipop graph $L_{m,n}$ is the graph obtained by joining the complete graph K_m with the path graph P_n by a bridge. In the present talk, we will give some algebraic properties of the edge ring of lollipop graph, $L_{m,n}$, in terms of its basic graphs, K_m and P_n.

This is a joint work with Farhad Rahmati‡.

1. INTRODUCTION

In recent years, studying algebraic invariants of monomial ideals associated with graphs, hypergraphs and simplicial complexes in terms of their combinatorial properties is a customary material. Let G be a simple graph with the vertex set $V_G = \{x_1, \ldots, x_n\}$ and the edge set E_G and let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. A squarefree monomial ideal generated by $x_i x_j$, where $\{x_i, x_j\}$ is an edge in G, is called edge ideal $I(G)$ of G. Let K_m be the complete graph with the vertex set $\{y_1, \ldots, y_m\}$ and P_n be the path graph with the vertex set $\{x_1, \ldots, x_n\}$. The (m,n)-lollipop graph $L_{m,n}$ is a graph obtained by joining the complete graph K_m to the path graph P_n by a bridge. Indeed, (m,n)-lollipop graph $L_{m,n}$ is the glued graph of P_{n+2} and K_m at K_2. The gluing is a natural graph operation that is defined in [4] and also has beneficial application in various fields such as chemical engineering, electrical engineering, computer engineering, applied mathematics and other sciences. In addition, gluing of graphs creates a new graph which its properties can be investigated by means of the initial graphs. The fact that path graphs and complete graphs are sufficiently well-known leads us naturally to the study of gluing of these graphs. In this paper, we will characterize the homological invariants of $I(L_{m,n})$ using the combinatorial data of the basic graphs.

2. MAIN RESULTS

Theorem 2.1. *For all $n \geq 2$, the i-th $\mathbb{N}$-graded Betti number of degree $i+1$ of the lollipop graph $L_{m,n}$ may be written as*

$$\beta_{i,i+1}(L_{m,n}) = \begin{cases} \beta_{i,i+1}(L_{m,n-1}) + 1 = i\binom{m}{i+1} + \binom{m-1}{i-1} + n - 1 & i = 1,2 \\ \beta_{i,i+1}(L_{m,n-1}) = i\binom{m}{i+1} + \binom{m-1}{i-1} & 3 \leq i \leq m \end{cases}$$

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Lemma 2.2. *For any $d \geq 2$, we have $\beta_{i,i+d}(L_{m,1}) = 0$.*

Theorem 2.3. *For all $2 \leq i \leq m$, i-th $\mathbb{N}$-graded Betti number of degree $i+2$ of the lollipop graph $L_{m,2}$ can be written as*

$$\beta_{i,i+2}(L_{m,2}) = (i-1)\binom{m-1}{i} + (i-2)\binom{m-1}{i-1}.$$

Theorem 2.4. *For all $2 \leq i \leq m+1$, i-th $\mathbb{N}$-graded Betti number of degree $i+2$ of the lollipop graph $L_{m,3}$ can be written as*

$$\beta_{i,i+2}(L_{m,3}) = (i-1)\binom{m}{i} + (i-2)\binom{m}{i-1} + (i-1)\binom{m-1}{i} + (i-2)\binom{m-1}{i-1}.$$

Corollary 2.5. *For any $n \geq 4$ and $2 \leq i \leq m+2$, i-th $\mathbb{N}$-graded Betti number of degree $i+2$ of the lollipop graph $L_{m,n}$ can be written as*

$$\beta_{i,i+2}(L_{m,n}) = \beta_{i,i+2}(L_{m,n-1}) + \beta_{i-1,i}(L_{m,n-3}) + \beta_{i-2,i-1}(L_{m,n-3}).$$

Lemma 2.6. *For all $d \geq 3$, i-th $\mathbb{N}$-graded Betti number of degree $i+d$ of the lollipop graph $L_{m,4}$ vanishes.*

Theorem 2.7. *For all $d \geq 3$, i-th $\mathbb{N}$-graded Betti number of degree $i+d$ of the lollipop graph $L_{m,5}$ can be written as*

$$\beta_{i,i+d}(L_{m,5}) = \begin{cases} (i-2)\binom{m-1}{i-1} + (2i-6)\binom{m-1}{i-2} + (i-4)\binom{m-1}{i-3} & d = 3 and\ 4 \leq i \leq m+2 \\ \binom{m-1}{2} & d = 3 andi = 3 \\ 0 & d \geq 4. \end{cases}$$

Theorem 2.8. *For all $d' \geq 3$ and $n \geq 6$, i-th $\mathbb{N}$-graded Betti number of degree $i+d'$ of the lollipop graph $L_{m,n}$ is non zero if and only if the following conditions are satisfied:*

1. *$n \geq 3d' - 4$,*
2. *$d' \leq i \leq m + 2d' - 2$.*

Moreover, if d is the maximum value of d' satisfied in condition (1), then condition (2) can be expressed as follows:

a. *if $n = 3k$ for some integer k then $d \leq i \leq m + 2d - 3$.*
b. *if $n = 3k+1$ for some integer k then $d \leq i \leq m + 2d - 2$.*
c. *if $n = 3k+2$ for some integer k then $d \leq i \leq m + 2d - 4$.*

If the above conditions are satisfied, then we have:

$$\beta_{i-1,i+d'}I(L_{m,n}) = \beta_{i-1,i+d'}I(L_{m,n-1}) + \beta_{i-2,i+d'-2}I(L_{m,n-3}) + \beta_{i-3,i+d'-3}I(L_{m,n-3}).$$

Theorem 2.9. *The projective dimension of the lollipop graph $L_{m,n}$ is independent of the characteristic of the chosen field and*

$$pd(L_{m,n}) = \begin{cases} m+2d-3 & n \equiv 0 mod\ 3 \\ m+2d-2 & n \equiv 1 mod\ 3 \\ m+2d-4 & n \equiv 2 mod\ 3, \end{cases}$$

where d is the maximum amount of d' satisfied in relation $n \geq 3d'-4$.

Corollary 2.10. *The Castelnuovo-Mumford regularity of the lollipop graph can be expressed as*

$$reg(L_{m,n}) = \begin{cases} \frac{n}{3}+1 & n \equiv 0 mod\ 3 \\ \frac{n+2}{3} & n \equiv 1 mod\ 3 \\ \frac{n+4}{3} & n \equiv 2 mod\ 3. \end{cases}$$

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A DENSE SUBGROUP OF TOPOLOGICAL FUNDAMENTAL GROUP OF QUOTIENT SPACES

HAMID TORABI

ABSTRACT. In this talk, we show that the image of the topological fundamental group of a given space X is dense in the topological fundamental group of the quotient space X/A under the induced homomorphism of the quotient map, where A is a suitable subspace of X with some conditions on X. Also, we give some applications to find out some properties for $\pi_1^{top}(X/A,*)$. In particular, we give some conditions under which $\pi_1^{top}(X/A,*)$ is an indiscrete topological group.

1. INTRODUCTION

Let X be a topological space and $\sim$ be an equivalent relation on X. Then one can consider the quotient topological space $X/\sim$ and the quotient map $p : X \longrightarrow X/\sim$. By applying the fundamental group functor on p there exists the induced homomorphism

$$p_* : \pi_1(X) \longrightarrow \pi_1(X/\sim).$$

It seems interesting to determine the image of the above homomorphism. Recently, Calcut, Gompf, and Mccarthy [2] proved that if X is a locally path connected topological space partitioned into connected subsets and if the associated quotient space $X/\sim$ is semi-locally simply connected, then the induced homomorphism, p_*, of fundamental groups is surjective for each choice of base point $x \in X$. They inspired the problem by one of Arnold's problems on orbit spaces of vector fields on manifolds [2, Section 3.6].

D. Biss [1] introduced the topological fundamental group $\pi_1^{top}(X,x)$ of a based space (X,x) as the fundamental group $\pi_1(X,x)$ with the quotient topology of the loop space $\Omega(X,x)$ with respect to the canonical function $\pi : \Omega(X,x) \longrightarrow \pi_1(X,x)$ identifying path components. It is known that this construction gives rise a homotopy invariant functor $\pi_1^{top} : htop_* \longrightarrow QTG$ from the homotopy category of based spaces to the category of quasitopological groups and continuous homomorphism [3]. By applying the above functor on the quotient map $p : X \longrightarrow X/\sim$, we have a continuous homomorphism $p_* : \pi_1^{top}(X) \longrightarrow \pi_1^{top}(X/\sim)$. For a pointed topological space (X,x), by a path we mean a continuous map $\alpha : [0,1] \longrightarrow X$. The points $\alpha(0)$ and $\alpha(1)$ are called the initial point and the terminal point of α, respectively. A loop α is a path with $\alpha(0) = \alpha(1)$. For a path $\alpha : [0,1] \longrightarrow X$, α^{-1} denotes a path such that $\alpha^{-1}(t) = \alpha(1-t)$, for all $t \in [0,1]$. Denote $[0,1]$ by I, two paths $\alpha, \beta : I \longrightarrow X$ with the same initial and terminal points are called homotopic relative to end

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points if there exists a continuous map $F : I \times I \longrightarrow X$ such that

$$F(t,s) = \begin{cases} \alpha(t) & s = 0 \\ \beta(t) & s = 1 \\ \alpha(0) = \beta(0) & t = 0 \\ \alpha(1) = \beta(1) & t = 1. \end{cases}$$

The homotopy class containing a path α is denoted by $[\alpha]$. Since most of the homotopies that appear in this paper have this property and end points are the same, we drop the term "relative homotopy" for simplicity. For paths $\alpha, \beta : I \longrightarrow X$ with $\alpha(1) = \beta(0)$, $\alpha * \beta$ denotes the concatenation of α and β, that is, a path from I to X such that $(\alpha * \beta)(t) = \alpha(2t)$, for $0 \leq t \leq 1/2$ and $(\alpha * \beta)(t) = \beta(2t-1)$, for $1/2 \leq t \leq 1$.

For a space (X,x), let $\Omega(X,x)$ be the space of based maps from I to X with the compact-open topology. A subbase for this topology consists of neighborhoods of the form $\langle K, U \rangle = \{\gamma \in \Omega(X,x) \mid \gamma(K) \subseteq U\}$, where $K \subseteq I$ is compact and U is open in X. When X is path connected and the basepoint is clear, we just write $\Omega(X)$ and we will consistently denote the constant path at x by e_x. The topological fundamental group of a pointed space (X,x) may be described as the usual fundamental group $\pi_1(X,x)$ with the quotient topology with respect to the canonical map $\Omega(X,x) \longrightarrow \pi_1(X,x)$ identifying homotopy classes of loops, denoted by $\pi_1^{top}(X,x)$. A basic account of topological fundamental groups may be found in [1], [3].

2. Main Results

Theorem 2.1. *Let A be an open subset of X such that $\overline{A}$ is path connected, then for each $a \in A$ the image of p_* is dense in $\pi_1^{top}(X/A,*)$ i.e:*

$$\overline{p_* \pi_1^{top}(X,a)} = \pi_1^{top}(X/A,*).$$

Definition 2.2. *Let X be a topological space and $A_1, A_2, ..., A_n$ be any subsets of X, $n \in \mathbb{N}$. By the quotient space $X/(A_1, ..., A_n)$ we mean the quotient space obtained from X by identifying each of the sets A_i to a point. Also, we denote the associated quotient map by $p : X \longrightarrow X/(A_1, A_2, ..., A_n)$.*

Corollary 2.3. *Let $A_1, A_2, ..., A_n$ be open subsets of a path connected space X such that the A_i's are path connected for every $i = 1, 2, ..., n$. Then for every $a \in \bigcup_{i=1}^{n} A_i$,*

$$p_* \pi_1^{top}(X,a) = \pi_1^{top}(X/(A_1, A_2, ..., A_n), *).$$

Corollary 2.4. *Let $A_1, A_2, ..., A_n$ be open subsets of a connected, locally path connected space X such that the A_i's are path connected for every $i = 1, 2, ..., n$. If $X/(A_1, A_2, ..., A_n)$ is semi-locally simply connected, then for each $a \in \bigcup_{i=1}^{n} A_i$,*

$$p_* : \pi_1^{top}(X,a) \longrightarrow \pi_1^{top}(X/(A_1, A_2, ..., A_n, *)$$

is an epimorphism.

In the following example we show that with assumption of previous Theorem p_* is not necessarily onto.

Examples 2.5. *Let $A_n = \{1/(2n-1), 1/2n\} \times [0, 1+1/2n] \bigcup [1/2n, 1/2n-1] \times \{1+1/2n\}$ for each $n \in \mathbb{N}$. Consider $X = (\bigcup_{n\in\mathbb{N}} A_n) \bigcup \{0\} \times [0,1] \bigcup [0,1] \times \{0\}$ with $a = (0,0)$ as the base point and $A = \{(x,y) \in X \mid y < 1\}$. A is an open subset of X with path connected closure. Let $I_n = (1/2+1/2(n+1), 1/2+1/2n]$ and f_n be a homeomorphism from I_n to $A_n - \{(1/2n, 0)\}$ for every $n \in \mathbb{N}$. Define $f : I \longrightarrow X$ by*

$$f(t) = \begin{cases} \text{the point } (0,2t) & t \in [0,1/2], \\ f_n(t) & t \in I_n. \end{cases}$$

We claim that $\alpha = p \circ f$ is a loop in X/A at $*$. It suffices to show that α is continuous on $t = 1/2$ and boundary points of I_n's since f is continuous on $[0,1/2)$ and by gluing lemma on $\bigcup int(I_n)$. Since α is locally constant at $t = 1/2+1/2n$ for each $n \in \mathbb{N}$, α is continuous at boundary points of I_n. For each open neighborhood G of $f(1/2) = (0,1)$ in X, there exists $n_0 \in \mathbb{N}$ such that G contains $A_n \bigcap A^c$ for $n > n_0$. Therefore continuity at $t = 1/2$ follows from $\alpha(1/2) \in \overline{\{*\}}$. Now let $B \subseteq \mathbb{N}$ and define

$$g_B(t) = \begin{cases} (p \circ f)(t) & t \in \bigcup_{m\in B} I_m, \\ * & otherwise. \end{cases}$$

Then g_B is continuous and for $B_1, B_2 \subseteq \mathbb{N}$ such that $B_1 \neq B_2$, $[g_{B_1}] \neq [g_{B_2}]$ which implies that $\pi_1(X/A, *)$ is uncountable. But by compactness of I, a given path in X can traverse finitely many of the A_n's and therefore $\pi_1(X,a)$ is a free group on countably many generators which implies that p does not induce a surjection of fundamental groups.

3. Some Applications

After that Biss [1] equipped the fundamental group of a topological space and named it topological fundamental group, Fabel [5] showed that topological fundamental groups can distinguish spaces with isomorphic fundamental groups. Hence studying the topology of fundamental groups seems important.

By $(X, A_1, A_2, ..., A_n)$ we mean an $(n+1)$-*tuple* of spaces where the A_i's are open subsets of X with path connected closure.

Theorem 3.1. *For an $(n+1)$-tuple of spaces $(X, A_1, A_2, ..., A_n)$, if X is simply connected, then $\pi_1^{top}(X/(A_1, A_2, ..., A_n), *)$ is an indiscrete topological group.*

Theorem 3.2. *For an $(n+1)$-tuple of spaces $(X, A_1, A_2, ..., A_n)$, if $\pi_1^{top}(X,a)$ is compact and $\pi_1^{top}(X/(A_1, A_2, ..., A_n), *)$ is Hausdorff, then p_* is an epimorphism and so is a closed quotient map.*

Theorem 3.3. *For an $(n+1)$-tuple of spaces $(X, A_1, A_2, ..., A_n)$, if $\pi_1^{top}(X,a)$ is compact and $\pi_1^{top}(X/(A_1, A_2, ..., A_n), *)$ is Hausdorff, then either $\pi_1^{top}(X/(A_1, A_2, ..., A_n), *)$ is a discrete topological group or uncountable.*

Corollary 3.4. *For an $(n+1)$-tuple of spaces $(X, A_1, A_2, ..., A_n)$, if $\pi_1^{top}(X,a)$ is a compact, countable quasitopological group, then either $X/(A_1, A_2, ..., A_n)$ is semi-locally simply connected or $\pi_1^{top}(X/(A_1, A_2, ..., A_n), *)$ is not Hausdorff.*

Corollary 3.5. *For an* $(n+1)$*-tuple of spaces* $(X,A_1,A_2,...,A_n)$*, if* $\pi_1^{top}(X,a)$ *is finite and* $\pi_1^{top}(X/(A_1,A_2,\ldots,A_n\ ),*)$ *is Hausdorff, then* $\pi_1^{top}(X/(A_1,A_2,\ldots,A_n),*)$ *is finite and* $X/(A_1,A_2,\ldots,A_n)$ *is semi-locally simply connected.*

Definition 3.6. (cf. [4]) *A topological space is said to be* **irreducible** *if it cannot be expressed as a union of two proper closed subsets. Equivalently, it is irreducible if each of its nonempty open subsets is dense. A space is called* **semi-irreducible** *if every disjoint family of its nonempty open subsets is finite.*

A sober space is a topological space X such that every irreducible closed subset of X is the closure of exactly one point of X ([4]). For example every Hausdorff space is sober space, since the only irreducible subsets are points. In [4] it is proved that a sober space is semi-irreducible if and only if it has a finite dense subset.

Corollary 3.7. *For an* $(n+1)$*-tuple of spaces* $(X,A_1,A_2,...,A_n)$*, if* $\pi_1^{top}(X,a)$ *is finite and* $\pi_1^{top}(X/(A_1,A_2,...,A_n\ ),*)$ *is a sober topological space, then* $\pi_1^{top}(X/(A_1,A_2,...,A_n\ ),*)$ *is semi-irreducible.*

Proof. Note that since $\pi_1^{top}(X/(A_1,A_2,...,A_n),*)$ has a finite dense subset, the result holds. □

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ON THE SCOTT CLOSED RETRACTS OF AUTOMATA ON DIRECTED COMPLETE POSETS

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ABSTRACT. Automata theory is an exciting branch of computer science and discrete mathematics. The algebraic theory of a particular type of automata (without outputs) is nothing but the actions of a free monoid or a free semigroup on a set (of states). On the other hand, domain theory, which studies directed complete partially ordered sets was introduced by Scott and has grown into a respected field on the borderline between mathematics and computer science. In this talk, combining the above two notions, we consider automata on dcpo and study the algebraic notion of retract with respect to class of monomorphisms for automata on directed complete posets.

This is a joint work with Mohammad Mehdi Ebrahimi‡ and Mojgan Mahmoudi††.

1. INTRODUCTION

First we recall some preliminaries needed in the sequel. The reader can find more details in [1], [2], [3], [4], [5]. Let **Pos** denote the category of all partially ordered sets (posets) with order preserving (monotone) maps between them. A subset D of a partially ordered set is called *directed*, denoted by $D \subseteq^d P$, if for every $a, b \in D$ there exists $c \in D$ such that $a, b \leq c$; and P is called *directed complete*, or briefly a *dcpo*, if for every $D \subseteq^d P$, the directed join $\bigvee^d D$ exists in P.

A *dcpo map* or a *continuous map* $f : P \to Q$ between dcpo's is a map with the property that for every $D \subseteq^d P$, $f(D)$ is a directed subset of Q and $f(\bigvee^d D) = \bigvee^d f(D)$. Thus we have the category **Dcpo** of all dcpo's with continuous maps between them.

A *dcpo-monoid* (*group*) is a monoid (group) which is also a dcpo whose binary operation is a continuous map.

Recall that a (*right*) *S-act* or *S-set* for a monoid S is a set A equipped with an *action* $A \times S \to A$, $(a, s) \mapsto as$, such that $ae = a$ (e is the identity element of S) and $a(st) = (as)t$, for all $a \in A$ and $s, t \in S$. Let **Act**-S denote the category of all S-acts with action preserving maps ($f : A \to B$ with $f(as) = f(a)s$, for all $a \in A, s \in S$). Let A be an S-act. An element $a \in A$ is called a *zero, fixed,* or *a trap element* if $as = a$, for all $s \in S$.

For a dcpo-monoid S, a (*right*) *S-dcpo* is a dcpo A which is also an S-act whose action $\lambda : A \times S \to A$ is a continuous map.

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So, by an *S-dcpo map* between S-dcpo's, we mean a map $f : A \to B$ which is both continuous and action preserving. We denote the categories of all S-dcpo's and S-dcpo maps between them by **Dcpo**-S.

Definition 1.1. *A morphism $f : A \to B$ in all the categories investigate in this paper is called order embedding, or briefly embedding, provided that for all $x, y \in A$, $f(x) \leq f(y)$ iff $x \leq y$.*

2. Retracts in Dcpo-S and sub categories of Dcpo-S with respect to Scott Closed Embeddings

In this section we investigate the concept of retracts in the category **Dcpo**-S and sub categories of **Dcpo**-S with respect to scott closed embeddings.

Definition 2.1. *A possibly empty subset A of a dcpo B is said to be scott closed in B if for each $a \in A$ and $b \in B$ with $b \leq a$ we have $b \in A$ (A is down closed in B) and for every $D \subseteq^d A, \bigvee^d D \in A$.*

Definition 2.2. *An embedding $f : A \to B$ is said to be scott closed embedding or sc-embedding if A is scott closed in B.*

Remark 2.3. *An S-dcpo map $f : A \to B$ is sc-embedding if and only if it is embedding and A is down closed in B.*

Definition 2.4. *Let B be an S-dcpo and A be a none-empty subset of B. A is called a sub S-dcpo of B if A is a sub dcpo and subact of B. Also, B is called an extension of A.*

Definition 2.5. *Suppose A and B are S-dcpo's and B is an extension of A and consider $i : A \to B$. An S-dcpo A is said to be a retract of B if there exists a morphism $f : B \to A$ such that $f \circ i = id_A$, in which case f is said to be a retraction. Also A is called an absolute retract if it is a retract of each of its extensions.*

Recall that for a class $\mathscr{M}$ of monomorphisms in a category $\mathscr{C}$, an object $A \in \mathscr{C}$ is called $\mathscr{M}$*-injective* if for each $\mathscr{M}$*-morphism* $g : B \to C$ and morphism $f : B \to A$ there exists a morphism $k : C \to A$ such that $kg = f$. That is, the following diagram is commutative:

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[dl, "k"] \\ A & \end{tikzcd}$$

In this paper we consider $\mathscr{M}$ to be the class of all scott closed embeddings in the appropriate categories. Thus, $\mathscr{M}$-injective is simply called sc-*injective*.

Lemma 2.6. *If A is a sc-absolute retract in* **Dcpo**-S *then it has zero top element.*

Theorem 2.7. *Let S be a dcpo monoid with any one of the following properties :*
(1) *S is a dcpo group.*
(2) $\perp_S = e_S$.
(3) *S is right zero with top element.*
Then, for object C in **Dcpo**-S, *the following conditions are equivalent:*
(i) *C has a zero top element.*
(ii) *C is sc-injective.*
(iii) *C is absolute retract with respect to sc-embeddings.*

Remark 2.8. *In the cases of Theorem 2.7,* **Dcpo**-*S has enough injective objects with respect to sc-embeddings.*

Definition 2.9. *An S-dcpo A is called reversible if for all $a \in A$ and $s \in S$, there exists $t \in S$ such that $ast = a$. So, we have the category* **R**-**Dcpo**-*S of all reversible S-dcpo's and S-dcpo maps between them.*

Theorem 2.10. *For object C in* **R**-**Dcpo**-*S the following conditions are equivalent:*
(i) *C has a zero top element.*
(ii) *C is sc-injective.*
(iii) *C is absolute retract with respect to sc-embeddings.*

Remark 2.11. *By Theorem 2.10,* **R**-**Dcpo**-*S has enough injective objects with respect to sc-embeddings.*

Theorem 2.12. *Consider the category $\mathscr{C}'$, whose class of objects are S-dcpo's with property that for each $A \in \mathscr{C}'$ we have, for all $a \in A$ and for all $s \in S$, $a \leq as$ and the class of morphisms is S-dcpo maps between them. Then for object $C \in \mathscr{C}'$, the following conditions are equivalent:*
(i) *C has a zero top element.*
(ii) *C is sc-injective.*
(iii) *C is absolute retract with recpect to sc-embeddings.*

Definition 2.13. *Let B be an S-dcpo. A sub S-dcpo A of B is called a filter, if for each $a \in A$ and $s \in S$, $as \in A$ implies $a \in A$.*

Remark 2.14. *We denote the class of all embeddings $f : A \to B$ in* **Dcpo**-*S such that A is scott closed in B and A is a filter of B by $\mathscr{M}'$.*

Theorem 2.15. *For object C in* **Dcpo**-*S, the following conditions are equivalent:*
(i) *C has a zero top element.*
(ii) *C is $\mathscr{M}'$-injective.*
(iii) *C is $\mathscr{M}'$-absolute retract.*

Remark 2.16. *By Theorem 2.15,* **Dcpo**-*S has $\mathscr{M}'$-enough injective objects.*

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MV-ALGEBRAS AND RELATED STRUCTURES

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ABSTRACT. In this talk, the variety of MV-algebras is verified and the tight relation between MV-algebras, semirings and lattice ordered groups are shown.

This is a joint work with R.A. Borzooei[‡].

1. INTRODUCTION

In [4], C. C. Chang devised MV-algebras to study many-valued logics, introduced by Jan Łukasiewicz in 1920. In particular, MV-algebras form the algebraic semantics of Łukasiewicz logic. This structure is closely related to other algebraic structures such as lattice ordered groups, BCK-algebras, BL-algebras and semirings. One of the most remarkable properties of MV-algebras is their tight relation with lattice-ordered groups. In fact, Muncici [9], proved that the category of MV-algebras, with morphisms corresponding to object homomorphisms, is equivalent to the category of Abelian lattice-ordered groups with strong unit with morphisms corresponding to homomorphisms preserving the strong unit, also, he showed that MV-algebras are categorically equivalent to bounded commutative BCK-algebras (see [8]). The main purpose of this paper is to verify the properties of this algebra and to show the relation between MV-algebras and lattice ordered groups. Then we see that any MV-algebra equation holding in the standard MV-algebra over the interval $[0,1]$ will hold in every MV-algebra. Algebraically, this means that the standard MV-algebra generates the variety of all MV-algebras (see [5, 10]). Also, the concepts of good sequence and change monoid of an MV-algebra are introduced and they are used to construct a lattice ordered group. Finally, Mundici functors between category of MV-algebras and lattice ordered groups with strong units are introduced and it is shown that these categories are equivalent.

2. MV-ALGEBRAS

Definition 2.1. [4] *An MV-algebra is an algebra $(M,\oplus,',0)$ of type $(2,1,0)$, where $(M,\oplus,0)$ is a commutative monoid with neutral element 0 and for all $x,y\in M$:*
(i) $x''=x$;
(ii) $x\oplus 1=1$, *where* $1=0'$;
(iii) $x\oplus(x\oplus y')'=y\oplus(y\oplus x')'$.

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In every MV-algebra $(M, \oplus, ', 0)$*, we can define the following further operations:*

$$x \odot y = (x' \oplus y')', \ \ x \ominus y = (x' \oplus y)', \ \ x \to y = x' \oplus y$$

Proposition 2.2. [7] *Let* $(A, \oplus, ', 0)$ *be an MV-algebra. Then* $A^{\vee\odot} = (A, \vee, \odot, 0, 1)$ *and* $A^{\wedge\oplus} = (A, \wedge, \oplus, 1, 0)$ *are semiring. Moreover, the involution* $' : A \to A$ *is an isomorphism between them.*

Definition 2.3. [7] *An MV-semiring is a commutative, additively idempotent semiring* $(A, \vee, ., 0, 1)$ *for which there exists a map* $* : A \to A$*, called the negation, satisfying the following conditions:*
(i) $a.b = 0$ *if and only if* $b \leq a^*$ *(where* $a \leq b$ *if and only if* $a \vee b = b$*) for all* $a, b \in A$*;*
(ii) $a \vee b = (a^*.(a^*b)^*)^*$ *for all* $a, b \in A$*.*

Proposition 2.4. [7] *For any MV-algebra* $(A, \oplus, ', 0)$*, both the semiring reducts* $A^{\vee\odot}$ *and* $A^{\wedge\oplus}$ *are MV-semirings. Conversely, if* $(A, \vee, ., 0, 1)$ *is an MV-semiring with negation* $*$*, the structure* $(A, \oplus, *, 0)$ *is an MV-algebra.*

In each MV-algebra $(A, \oplus, 0)$ the relation $x \leq y$ if and only if $x' \oplus y = 1$ is a order relation and $(A, \leq)$ is a lattice. It can be easily shown that $x \vee y = x \oplus (x \oplus y')'$ and $x \wedge y = (x' \vee y')'$, for all $x, y \in A$.

Theorem 2.5. *Every MV-algebra A is a subdirect product of MV-chains.*

Theorem 2.6. *(Chang Completeness Theorem) An equation holds in* $[0, 1]$ *if and only if it holds in every MV-algebra.*

When we want to show that an identity holds on an MV-algebra A, it suffices to consider A is an MV-chain. For example, since any chain is a distributive lattice, then by Theorem 2.6 we conclude that each MV-algebra is a distributive lattice.

3. RELATION BETWEEN l-GROUPS AND MV-ALGEBRAS

Definition 3.1. [1, 2] A partially ordered abelian group is an abelian group $(G, +, -, 0)$ endowed with a partial order relation $\leq$ that is compatible with addition: in other words, $\leq$ has the following translation invariance property, for all $x, y, t \in G$, if $x \leq y$, then $x + t \leq y + t$.

For each element x of an l-group G, we define $x^+ = 0 \vee x$, $x^- = 0 \vee -x$ and $|x| = x^+ + x^- = (x \vee -x)$. A strong unit u of G is an element $0 \leq u \in G$ such that for each $x \in G$ there is an integer $n \in \mathbb{N}_0$ such that $|x| \leq nu$.

Example 3.2. [1] 1- Clearly, $(\mathbb{R}, +, 0)$ and $(\mathbb{Z}, +, 0)$ are l-groups.

2- Let K be a field and ν be a K-valuation. Then $\Gamma_\nu = \nu(K^*)$ is a totally ordered abelian group, called the value group of ν.

3- A Bezout domain is an integral domain in which the sum of two principal ideals is again a principal ideal (Any principal ideal domain (PID) is a Bzout domain, but a Bzout domain need not be Noetherian ring, so it could have non-finitely generated ideals). Every l-group is the group of divisibility of some Bezout domain.

Let G be an l-group. For any element $u \in G$, $u > 0$, we let $[0, u] = \{x \in G | \ 0 \leq x \leq u\}$ and for each $x, y \in [0, u]$, $x \oplus y = u \wedge (x + y)$, $\ x' = u - x$. Then the structure $([0, u], \oplus, ', 0)$ is an MV-algebra. It is denoted by $\Gamma(G, u)$.

Example 3.3. [10] Let $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{Z}$ be groups of reals, rationals, integers, with the natural order. In particular case when $G=\mathbb{R}$, $\Gamma(G,1)$ coincides with the MV-algebra $[0,1]$. We also have $[0,1]\cap\mathbb{Q}=\Gamma(\mathbb{Q},1)$. Further, for each integer $2\leq n$, $L_n=\Gamma(\mathbb{Z}1/(n-1),1)$ where $\mathbb{Z}1/(n-1)=\{z/(n-1)|\, z\in\mathbb{Z}\}$ In particular, the boolean algebra $\{0,1\}$=L_2 is the same as $\Gamma(\mathbb{Z},1)$.

From now on, $\mathscr{A}$ will denote the category whose objects are pairs (G,u) with G an l-group and u a distinguished strong unit of G and whose morphisms are unital l-morphisms. Moreover, $\mathscr{MV}$ is the category of MV-algebras.

Theorem 3.4. [10] Γ *is a functor from* $\mathscr{A}$ *into* $\mathscr{MV}$.

Definition 3.5. [6] A sequence $a=(a_1,a_2,...)$ of elements of an arbitrary MV-algebra A is said to be good if and only if for each $i\in\mathbb{N}$, $a_i\oplus a_{i+1}=a_i$ and there is an integer n such that $a_r=0$ for all $n<r$. Instead of $a=(a_1,a_2,...)$, we shall often write, more concisely, $\mathbf{a}=(a_1,a_2,...,a_n)$. Thus we have identical good sequences $(a_1,a_2,...,a_n)=(a_1,a_2,...,a_n,0^m)$, where 0^m denotes an m-tuple of zeros. For each $a\in A$, the good sequence $(a,0,...,0,...)$ will be denoted by (a).

Proposition 3.6. *Suppose that* $A\subseteq\Pi_{i\in I}A_i$ *is the subdirect product of a family* $\{A_i\}_{i\in I}$ *of* MV*-algebras. A sequence* $\mathbf{a}=(a_1,a_2,...,a_n)$ *of elements of* A *is a good sequence if and only if for each* $i\in I$ *the sequence* $(\pi_i(a_1),\pi_i(a_2),...,\pi_i(a_n))$ *is a good sequence in* A_i*, and there is an integer* $0\leq n_0$ *such that whenever* $n_0<n$ *then for all* $i\in I$*,* $\pi_i(a_n)=0$.

Lemma 3.7. *Let* A *be an* MV*-algebra. If* $\mathbf{a}=(a_1,a_2,...,a_n,...)$ *and* $\mathbf{b}=(b_1,b_2,...,b_n,...)$ *are good sequences of* A*, then so is the sequence* $\mathbf{c}=(c_1,c_2,...,c_n,...)$ *given by* $c_n=a_n\vee b_n$ *for each* n.

Definition 3.8. For any two good sequences $\mathbf{a}=(a_1,a_2,...,a_n)$ and $\mathbf{b}=(b_1,b_2,...,b_n)$ their sum $c=a+b$ is defined by $\mathbf{c}=(c_1,c_2,...)$ where for all $i\in\mathbb{N}$

$$c_i=a_i\oplus(a_{i-1}\odot b_1)\oplus(a_{i-2}\odot b_2)\oplus....\oplus(a_1\odot b_{i_1})\oplus b_i. \tag{1}$$

Definition 3.9. [6] Let A be an MV-algebra and M_A be the set of all good sequence on A. Then $(M_A,+,0)$ is a commutative monoid.

Given any two good sequence $\mathbf{a}=(a_1,...,a_n)$ and $\mathbf{b}=(b_1,...,b_m)$ of A, we write $\mathbf{a}\leq\mathbf{b}$ if and only if $a_i\leq b_i$, for all i. It can be easily shown that $\mathbf{a}\leq\mathbf{b}$ if and only if there exists a good sequence $\mathbf{c}$ such that $\mathbf{b}=\mathbf{a}+\mathbf{c}$. Moreover, by (i), $\mathbf{c}$ is unique and $\mathbf{c}=\mathbf{b}+(-\mathbf{a})$. Define a relation θ on $M_A\times M_A$ by $(\mathbf{a},\mathbf{b})\theta(\mathbf{c},\mathbf{d})$ if and only if $\mathbf{a}+\mathbf{d}=\mathbf{b}+\mathbf{c}$. The θ is a equivalence relation. Let G_A be the set of all equivalence classes of $M_A\times M_A$ with respect to θ. Then $(G_A,+,-,0)$ is a commutative groups, where $[\mathbf{a},\mathbf{b}]+[\mathbf{c},\mathbf{d}]=[\mathbf{a}+\mathbf{c},\mathbf{b}+\mathbf{d}]$ and $-[\mathbf{a},\mathbf{b}]=[\mathbf{b},\mathbf{a}]$ It is called *enveloping group* of A (see [10]).

Definition 3.10. [6] Let A be an MV-algebra, and $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\in M_A$. We say that the equivalence class $[c,d]$ dominates the equivalence class $[a,b]$, in symbols, $[\mathbf{a},\mathbf{b}]\preceq[\mathbf{c},\mathbf{d}]$ if $[c,d]-[a,b]=[e,(0)]$ for some good sequence $e\in M_A$. Equivalently, $[\mathbf{a},\mathbf{b}]\preceq[\mathbf{c},\mathbf{d}]$ if and only if $\mathbf{a}+\mathbf{d}\leq\mathbf{c}+\mathbf{b}$, where $\leq$ is the partial order of M_A.

Definition 3.11. [6] Let A be an MV-algebra. The relation $\preceq$ is a partial order, making G_A into an l-group. Specially, for any two pairs of good sequences $(\mathbf{a},\mathbf{b})$ and $(\mathbf{c},\mathbf{d})$

(1) $[\mathbf{a},\mathbf{b}]\vee[\mathbf{c},\mathbf{d}]=((\mathbf{a}+\mathbf{d})\vee(\mathbf{c}+\mathbf{b}),\mathbf{b}+\mathbf{d})$,

(2) Similarly, the infimum is given by $[\mathbf{a},\mathbf{b}]\wedge[\mathbf{c},\mathbf{d}] = ((\mathbf{a}+\mathbf{d})\wedge(\mathbf{c}+\mathbf{b}),\mathbf{b}+\mathbf{d})$.

The l-group G_A with the above lattice order is called the *chang l-group* of the MV-algebra A.

Proposition 3.12. [8] *The element $u_A = [(1),(0)]$ is a strong unit of the l-group G_A. Moreover, the correspondence $a \mapsto \varphi(a) = [(a),(0)]$ defines an isomorphism from the MV-algebra A onto the MV-algebra $\Gamma(G_A,u_A)$.*

Proposition 3.13. [10] *Let A and B be two MV-algebras and $h : A \to B$ be a homomorphism. For any good sequence $\mathbf{a} = (a_1,a_2,...,a_n)$ of M_A, $(h(a_1),h(a_2),...,h(a_n))$ is a good sequence of B. Suppose that $h^* : M_A \to M_B$ is defined by $h^*(\mathbf{a}) = (h(a_1),h(a_2),...,h(a_n))$. It follows from definition that, $h^*(\mathbf{a}+\mathbf{b}) = h^*(\mathbf{a})+h^*(\mathbf{b})$, $h^*(\mathbf{a}\vee\mathbf{b}) = h^*(\mathbf{a})\vee h^*(\mathbf{b})$, $h^*(\mathbf{a}\wedge\mathbf{b}) = h^*(\mathbf{a})\wedge h^*(\mathbf{b})$, for all $\mathbf{a},\mathbf{b}\in M_A$. So, h^* is both lattice and monoid homomorphism. Let us further define the map $h^\sharp : G_A \to G_B$ by $h^\sharp([\mathbf{a},\mathbf{b}]) = [h^*(\mathbf{a}),h^*(\mathbf{b})]$.*

Let u_A and u_B be the strong units of G_A and G_B given by Proposition 2.4.4. Then $h^\sharp : G_A \to G_B$ is a unital l-group homomorphism of $(G_A,+,u_A)$ to $(G_B,+,u_B)$. Now, consider that follows:

$$E(A) = (G_A,+,u_A), \;\; E(h) = h^\sharp$$

Theorem 3.14. [8] *E is a functor from the category of MV-algebras to the category of l-groups with distinguished strong unit. The composite functor ΓE is naturally equivalent to the identity functor of $\mathscr{MV}$ (category of MV-algebras). In other words, for all MV-algebras A, B and homomorphism $h : A \to B$, we have a commutative diagram:*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \downarrow\varphi_A & & \downarrow\varphi_B \\ \Gamma(E(A)) & \xrightarrow{\Gamma(E(h))} & \Gamma(E(B)). \end{array}$$

Our next aim is to prove that the composite functor $E\Gamma$ is also naturally equivalent to the identity functor of the category $\mathscr{A}$.

Proposition 3.15. [6] *Suppose that $(G,+)$ is an l-group with order unit u and $A = \Gamma(G,u)$. For each $0 \le a \in G$ there is a good sequence $g(a) = (a_1,a_2,...,a_n)$ such that $a = a_1+a_2+...+a_n$.*

Proposition 3.16. [10] *Let $(G,+)$ be an l-group with order unit u and $g : G^+ \to M_{\Gamma(G,u)}$ be the map which was defined in the last lemma. Then $g(a+b) = g(a)+g(b)$, $g(a\vee b) = g(a)\vee g(b)$, $g(a\wedge b) = g(a)\wedge g(b)$ and $g(u) = u$, for all $a,b \in G$.*

Corollary 3.17. *For every $(G,u) \in \mathscr{A}$, let the map $\psi_{(G,u)} : G \to G_{\Gamma(G,u)}$ be defined by $\psi_{(G,u)}(a) = [g(a^+),g(a^-)]$, for all $a \in G$. It follows that $\psi_{(G,u)}$ is an l-group isomorphism of G onto $G_{\Gamma(G,u)}$ and $\psi_{(G,u)}(u) = [(u),(0)]$.*

Theorem 3.18. [11] *The composite functor $E\Gamma$ is naturally equivalent to the identity functor of the category $\mathscr{A}$ (unital l-group). In other words, for every two unital l-groups (G,u), (H,v) and unital l-homomorphism $f : (G,u) \to (H,v)$, we have a commutative diagram:*

$$\begin{array}{ccc} (G,u) & \xrightarrow{f} & (H,v) \\ \downarrow\psi_{(G,u)} & & \downarrow\psi_{(H,v)} \\ E(\Gamma((G,u))) & \xrightarrow{E(\Gamma(f))} & E(\Gamma((H,v))). \end{array}$$

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A GENERALIZATION OF ZERO-DIVISOR GRAPHS BASED ON THE RADICAL OF IDEALS

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ABSTRACT. Let R be a commutative ring and let $\sqrt{I}$ denote the radical of an ideal I of R. We define an undirected graph $\mathscr{G}_I(R)$ with vertices $\{x \in R \setminus \sqrt{I} \mid xy \in I$ for some $y \in R \setminus \sqrt{I}\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$. This is a connected subgraph of the Redmond's ideal-based zero-divisor graph $\Gamma_{\sqrt{I}}(R)$. We give conditions under which $\mathscr{G}_I(R)$ and $\Gamma_{\sqrt{I}}(R)$ are equal or isomorphic. We show that if I is a 2-absorbing ideal of R, then $\mathscr{G}_I(R) = \emptyset$ or $\mathscr{G}_I(R)$ is a complete bipartite graph.

This is a joint work with Hosein Fazaeli Moghimi‡.

1. Introduction

Let R be a commutative ring. . In [1], D.F. Anderson and P.S. Livingston introduced the zero-divisor graph of a commutative ring R with identity, denoted by $\Gamma(R)$, as the graph with vertices $Z(R) = Z(R) \setminus 0$, the set of nonzero zero-divisors of R, and two distinct vertices x and y are adjacent if and only if $xy = 0$. This concept due to I. Beck [3], who let all the elements of R be vertices of $\Gamma(R)$ and was mainly interested in colorings. S. P. Redmond [7] introduced the zero-divisor graph with respect to an ideal I of R, denoted by $\Gamma_I(R)$, as the graph with vertex set $\{x \in R \setminus I : xy \in I$ for some $y \in R \setminus I\}$, and two distinct vertices x and y are adjacent if and only if $xy \in I$. It is easily seen that $\mathscr{G}_I(R)$ is a subgraph of $\Gamma_{\sqrt{I}}(R)$. We show that if R is a finite ring, then $\mathscr{G}_I(R) = \Gamma_{\sqrt{I}}(R)$ (Corollary 2.5). If R is a ring such that every star subgraph of $\mathscr{G}_I(R)$ is finite, then $\mathscr{G}_I(R) \cong \Gamma_{\sqrt{I}}(R)$ (Theorem 2.6). If I is a quasi-primary (in particular primary or prime) ideal of R, then $\mathscr{G}_I(R) = \emptyset$ (Corollary 2.2), and if I is a 2-absorbing ideal of R, then $\mathscr{G}_I(R) = \emptyset$ or $\mathscr{G}_I(R)$ is a bipartite graph (Theorem 2.8). Finally, we show that $\mathscr{G}_I(R)$ is a connected graph with diameter at most 3 and girth at most 7 (Theorem 2.9 and Corollary 2.10).

2. main results

Let R be a ring. We recall that an ideal I of R is a radical ideal if $\sqrt{I} = I$. Also, a ring R is a prime ring if 0 is a radical ideal of R.

Proposition 2.1. *Let R be a ring and I be an ideal of R. Then* $\mathscr{G}_{\sqrt{I}}(R) = \Gamma_{\sqrt{I}}(R) = \Gamma(R/\sqrt{I})$. *In particular ,if R is a prime ring and* $I = 0$*, then* $\mathscr{G}_I(R) = \Gamma_I(R) = \Gamma(R)$.

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A proper ideal I of a ring R is called quasi-primary if $xy \in I$, for $x, y \in R$, implies that $x \in \sqrt{I}$ or $y \in \sqrt{I}$. Indeed, I is a quasi-primary ideal of R if and only if $\sqrt{I}$ is a prime ideal of R [5, Definition 2, p. 176]. Clearly every primary ideal is quasi-primary, but the converse is not true generally (See for example [6, Exercise 11(b) page 56]).

Corollary 2.2. *Let R be a ring. If I is a quasi-primary ideal, then $\mathscr{G}_{\sqrt{I}}(R) = \Gamma_{\sqrt{I}}(R) = \Gamma(R/\sqrt{I}) = \emptyset$. In particular, this equality holds when I is an ideal of an Artinian local ring R.*

The following example shows that $\mathscr{G}_I(R)$ is not necessarily equal to $\Gamma_{\sqrt{I}}(R)$.

Example 2.3. *Let $R = \mathbb{Z}$ and $I = 12\mathbb{Z}$. It is clear that I is not a quasi-primary radical ideal of R, since $\sqrt{I} = 6\mathbb{Z}$. Also 2 and 3 are vertices of both graphs $\mathscr{G}_I(R)$ and $\Gamma_{\sqrt{I}}(R)$, and are adjacent in $\Gamma_{\sqrt{I}}(R)$ while are not in $\mathscr{G}_I(R)$.*

It is clear that if $\mathscr{G}_I(R)$ is infinite, then so is $\Gamma_{\sqrt{I}}(R)$. Suppose that $\mathscr{G}_I(R)$ is a finite graph and x is a vertex of $\Gamma_{\sqrt{I}}(R)$. Let $y_1, \cdots, y_n$ be all vertices that are adjacent to x in $\Gamma_{\sqrt{I}}(R)$. Then for each $1 \leq i \leq n$, there exists a minimum positive integer k_i such that $(xy)^{k_i} \in I$. Now if $k = max\{k_i \mid 1 \leq i \leq n\}$, then by the assignments $x \mapsto x^k$ and $y_i \mapsto {y_i}^k$ for vertices and $\{x, y_i\} \mapsto \{x^k, {y_i}^k\}$ for edges, $\Gamma_{\sqrt{I}}(R)$ is embedded in the $\mathscr{G}_I(R)$. Thus we have the following:

Theorem 2.4. *Let R be a ring and I be an ideal of R. Then either both $\mathscr{G}_I(R)$ and $\Gamma_{\sqrt{I}}(R)$ are infinite or $\mathscr{G}_I(R) = \Gamma_{\sqrt{I}}(R)$.*

Corollary 2.5. *If R is a finite ring, then $\mathscr{G}_I(R) = \Gamma_{\sqrt{I}}(R)$.*

If R is an infinite ring such that $\sqrt{0}$ is a prime ideal of R, then by the proof of [8, Proposition1.3], $\mathscr{G}_0(R) \neq \emptyset$. Thus, in this case, $\mathscr{G}_0(R) \ncong \Gamma_{\sqrt{0}}(R)$. However, with an argument similar to that of Theorem 2.4, we have the following:

Theorem 2.6. *Let R be a ring. If every star subgraph of $\mathscr{G}_I(R)$ is finite, then $\mathscr{G}_I(R) \cong \Gamma_{\sqrt{I}}(R)$.*

In Corollary 2.2, we saw that for every quasi-primary ideal I of R, and in particular for every prime ideal I of R, $\mathscr{G}_I(R) = \emptyset$. Now suppose that I is a 2-absorbing ideal of R. The next Theorem gives a characterization of $\mathscr{G}_I(R)$. We recall that a nonzero proper ideal I of R is a 2-absorbing ideal of R if whenever $x, y, z \in R$ and $xyz \in I$, then $xy \in I$ or $xz \in I$ or $yz \in I$ [2].

Theorem 2.7. (cf. [2, Theorem 2.4]) *Let I be a 2-absorbing ideal of R. Then one of the following statements must hold:*

(1) *$\sqrt{I} = p$ is a prime ideal of R such that $p^2 \subseteq I$.*
(2) *$\sqrt{I} = p_1 \cap p_2$, $p_1 p_2 \subseteq I$, and $(\sqrt{I})^2 \subseteq I$ where p_1, p_2 are the only distinct prime ideals of R that are minimal over I.*

The complete bipartite graph, denoted by $K_{m,n}$, is the graph whose vertex set is the disjoint union of two sets, V_1 and V_2, satisfying $|V_1| = m$, $|V_2| = n$, and whose edge set is precisely $\{\{v_1, v_2\} | v_1 \in V_1$ and $v_2 \in V_2\}$.
If I is a 2-absorbing ideal of R and $\sqrt{I}$ is not a prime ideal, then by Theorem 2.7 $\sqrt{I} = p_1 \cap p_2$, $p_1 p_2 \subseteq I$, and $(\sqrt{I})^2 \subseteq I$ where p_1, p_2 are the only distinct prime ideals of R that are minimal over I. Now for $x, y \in R \setminus \sqrt{I}$ with $xy \in I$, we have $xy \in p_1$ and $xy \in p_2$. Since p_1

and p_2 are prime, we have $x \in p_1$ or $y \in p_1$ and $x \in p_2$ or $y \in p_2$ and $x, y \notin p_1 \cap p_2$. Without loss of generality, we may assume that $x \in p_1 \setminus p_2$ and $y \in p_2 \setminus p_1$. Thus we have:

Theorem 2.8. *Let R be a ring and I a 2-absorbing ideal of R. Then $\mathscr{G}_I(R) = \emptyset$ or $\mathscr{G}_I(R) \cong K_{m,n}$ where $|p_1 \setminus p_2| = m$, $|p_2 \setminus p_1| = n$.*

Recall that a graph is said to be connected if for each pair of distinct vertices x and y, there is a finite sequence of distinct vertices $x = x_1, \cdots, x_n = y$ such that each pair $\{x_i, x_{i+1}\}$ is an edge. Such a sequence is said to be a path and the distance, $d(x, y)$, between connected vertices x and y is the length of the shortest path connecting them. The diameter of a connected graph G, $diam(G)$, is the supremum of the distances between vertices (and let $d(x, y) = \infty$ if no such path exists).

Let I be an ideal of a ring R. Let x and y be distinct vertices of $\mathscr{G}_I(R)$. If $xy \in I$, then $x - y$ is a path in $\mathscr{G}_I(R)$. Let $xy \notin I$. Then there exist $a, b \in R \setminus \{\sqrt{I} \cup \{x, y\}\}$ such that $ax \in I$ and $by \in I$. If $a = b$, then $x - a - y$ is a path. If $a \neq b$ and $ab \in \sqrt{I}$, i.e. $a^n b^n \in I$ for some positive integer n, then $x - a^n - b^n - y$ is a path. If $a \neq b$ and $ab \notin \sqrt{I}$, then $x - ab - y$ is a path. Thus analogous to [1, Theorem 2.3] for $\Gamma(R)$ and [7, Theorem 2.4] for $\Gamma_I(R)$ we have:

Theorem 2.9. *Let I be an ideal of a ring R. Then $\mathscr{G}_I(R)$ is a connected graph and*

$$diam(\mathscr{G}_I(R)) \leq 3.$$

A cycle in a graph G is a path that begins and ends at the same vertex. The girth of G, written $gr(G)$, is the length of the shortest cycle in G (and $gr(G) = \infty$ if G has no cycles). For any undirected graph G, $gr(G) \leq 2diam(G) + 1$ if G contains a cycle [4, Proposition 1.3.2]. Thus we have:

Corollary 2.10. *Let I be an ideal of a ring R. If $\mathscr{G}_I(R)$ contains a cycle, then $gr(\mathscr{G}_I(R)) \leq 7$.*

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به نام خدا

با سلام و خیرمقدم به خانم ها و آقایان، ریاست و معاونین محترم دانشگاه، مهمانان مدعو به ویژه پروفسور گوتو و پروفسور زارزوالا، استادان و دانشجویان محترم. بسیار افتخار می کنم که آغاز بیست و چهارمین سمینار جبر ایران، که یکی از سمینارهای سالانهٔ انجمن ریاضی ایران است، را در دانشگاه خوارزمی اعلام نمایم. همان گونه که از اعلامیه ها مشهود است، این سمینار به قدردانی از خدمات علمی و پژوهشی استاد ارجمند جناب آقای دکتر حسین ذاکری، به مناسبت بازنشستگی و هفتادمین سالگی ایشان، اختصاص یافته است و بسیار سپاسگزار مهمانان و به ویژه مهمانان خارجی هستیم که دعوت ما را جهت این واقعه مبارک پذیرا شده اند.

قطعاً" برای شرکت کنندگان، به خصوص برای جوانانی که احتمالاً" تاکنون با آقای دکتر ذاکری درس نداشته اند، این سؤال اساسی مطرح است که انگیزهٔ این کار دانشگاه خوارزمی و به ویژه برگزار کنندگان سمینار در پس قدردانی از ایشان چیست.

برای ورود به بحث اجازه می خواهم ابتدا مقدمه ای بر سخن خود بیان کنم. در دنیای امروز، یعنی در دهه دوم قرن بیست و یک، هر روز، با پدیده ای جدید، کشفیاتی بدیع، و طوفانی از نوآوری ها مواجه می شویم. همه ما خواسته یا ناخواسته، با این پدیده های نوظهور روبه رو می شویم. گاهی خود را با برخی از تغییرات هماهنگ می کنیم و گاهی، به دلیل ناشناخته بودن آن ها و یا عدم آمادگی شخصی یا محیطی، با آن همگام نیستیم. ولی آن چه مسلم است این دگرگونی ها بر زندگی ما تاثیر عمیقی می گذارد. امروزه به مجموعه تمام آثار این دگرگونی ها "توسعه" اطلاق می شود. این همان مفهومی است که تمدن امروزی با اندازه-گیری هایی کشورهای دنیا را به "توسعه یافته"، "در حال توسعه" و "توسعه نیافته" افراز می کند. از همین منظر است که یک سؤال اساسی پیش رو داریم: در جامعه خود و در این قسمت از دنیا، «ما» و به ویژه «من» در کجای این توسعه قرار دارم؟ به عقیده من اگر بخواهیم خدمت و نقش یک شهروند را ارزیابی کنیم، باید میزان مشارکت وی را در گسترش "توسعه" جامعه پیرامونی خود ارزیابی کنیم.

اکنون ارزیابی خدمات استاد ارجمند آقای دکتر ذاکری را از این زاویه یعنی میزان مشارکت ایشان در توسعه علم ریاضی و به ویژه در ارتقای سطح دانش جبر جابه جایی، بررسی می کنیم. ایشان در سال ۱۳۵۳ از مؤسسه ریاضیات دکتر غلامحسین مصاحب، تحت آموزش های آن عالم توسعه طلب مرحوم دکتر غلامحسین مصاحب، فارغ التحصیل شدند و در همان سال به عنوان استادیار در دانشگاه خوارزمی (تربیت معلم آن زمان) به تدریس مشغول شدند. در سال ۱۳۵۸ به عنوان بورسیه به انگلستان اعزام گردیدند و زیر نظر پروفسور R. Y. Sharp به ادامه تحصیل مشغول شدند. از نگاه دانش امروز مسأله دکترای ایشان را این گونه می توان بیان کرد: هنگامی که به ساختار مدول پاکت انژکتیو یک حلقه گرنشتاین دقت می کنیم می بینیم آن مدول ساختمانی کسری دارد، به عبارت دیگر هر عضو آن یک کسر است. اولین سؤالی که مطرح می شود آن است که آیا بقیه مدول های موجود در تحلیل انژکتیو حلقه گرنشتاین چنین ساختاری دارند؟ ایشان در رساله خود ثابت کردند که تمامی مدول های موجود در تحلیل مینیمال انژکتیو حلقه گرنشتاین ساختار کسری دارند و آن کسرها را کسرهای تعمیم یافته نامیدند. تا آن جایی که من اطلاع دارم تاکنون از سوی بیش از ۸۰ مقاله معتبر به آن ارجاع شده است. ایشان پس از بازگشت از انگلستان با تعداد زیادی دانشجوی مشتاق و تشنه دوره دکتری، که اتفاقاً" به تازگی در کشور راه اندازی شده بود، مواجه شدند که الحق در هدایت آنان همت گماردند. ۲۵ دکترای ریاضی تربیت نمودند که عموماً" از استادان شاخص کشور هستند. تعداد پایان نامه های کارشناسی ارشد تحت هدایت ایشان بالغ بر ۷۰ تا است. این همان نکته ای است که می خواهم نتیجه بگیرم که نقش استاد در این طوفان مبارک یعنی "توسعه"، بسیار مثبت و ستودنی است.

اجازه می خواهم از این فرصت استفاده کنم و به نمایندگی از سوی تمامی دانشجویان ایشان، که بنده خود نیز یکی از آنان هستم، از ایشان تشکر کنم و به ایشان خسته نباشید بگویم و برایشان سلامتی و عمر طولانی آرزو نمایم. در پایان اجازه می خواهم از بسیاری از اشخاص حقیقی و حقوقی زیر تشکر خود را ابراز کنم: انجمن ریاضی ایران، مسؤلین محترم دانشگاه خوارزمی، پژوهشگاه دانش های بنیادی IPM، اعضای محترم کمیتهٔ علمی که با ارزیابی های دقیق خود ما را در انتخاب مقالات کمک شایان نمودند، اعضای کمیتهٔ اجرایی، و تمامی اشخاصی که مقالات خود را جهت سخنرانی برای سمینار فرستادند. در پایان از تمامی شما که دانشگاه خوارزمی را به عنوان میزبان انتخاب نمودید و در سمینار شرکت کردید تشکر می کنم.

محمد تقی دیبایی (دبیر کمیتهٔ علمی)

۲۱ آبان ۱۳۹۳

چکیده‌های مبسوط

بیست و چهارمین سمینار جبر

۲۱ و ۲۲ آبان ۱۳۹۳

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بیست و چهارمین سمینار جبر

۲۱ و ۲۲ آبان ۱۳۹۳

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